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# Wall-Effects on Pressure Fluctuations in Quasi-Incompressible and Compressible Turbulent Plane Channel Flow

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**The purpose of the present paper is to extend the analysis of turbulent correlations containing pressure fluctuations $p'$, separating $p'$ not only into rapid $p'_{(\mathrm{r})}$ and slow $p'_{(\mathrm{s})}$ parts (as is usually done in wall-bounded DNS computations), but further into volume (weakly inhomogeneous; $p'_{(\mathrm{r};\mathfrak{V})}$ and $p'_{(\mathrm{s};\mathfrak{V})}$) and surface (strongly inhomogeneous wall-echo; $p'_{(\mathrm{r};w)}$ and $p'_{(\mathrm{s};w)}$) terms. An analysis of the Poisson equations for $p'$ in compressible flow, suggests the decomposition into 10 terms, each of which is further separated into volume and wall-echo parts. A DNS processing module is developed which computes the above splittings for various correlations containing pressure fluctuations (redistribution, pressure diffusion, velocity/pressure-gradient). Results are presented for DNS computations of quasi-incompressible, subsonic and supersonic plane channel flows, and used for the *a priori* near-wall assessment of standard Reynolds-stress models.**

## I. Introduction

Understanding the physics of turbulent fluctuations of pressure $p'$ is of major importance, not only because of their direct implication in noise[1,2] and excitation of immersed solid surfaces,[3] but also because they appear in correlations present in the transport equations for the Reynolds-stresses and the dissipation tensor.[4] Traditionally the analysis of $p'$ is based on the Poisson equation for the fluctuating pressure,[5] which, at the incompressible flow limit ($\rho \cong \mathrm{const} \cong \bar{\rho}\,;\forall t,x$ and $\mu \cong \mathrm{const} \cong \bar{\mu}\,;\forall t,x$), reads

$$\frac{1}{\bar{\rho}}\nabla^2 p' \cong \frac{1}{\bar{\rho}}\nabla^2(p'_{\mathrm{r}}+p'_{\mathrm{s}}) = \underbrace{-2\frac{\partial u'_k}{\partial x_\ell}\frac{\partial \bar{u}_\ell}{\partial x_k}}_{\text{rapid}}\ \underbrace{-\frac{\partial^2}{\partial x_\ell \partial x_k}(u'_k u'_\ell - \overline{u'_k u'_\ell})}_{\text{slow}} \quad ; \quad [M \longrightarrow 0\ ;\ \rho,\ \mu \cong \mathrm{const}] \tag{1}$$

This equation can be extended to compressible flow[6–10] but in this case the unsteady terms (*eg* $\partial^2_{tt}\rho'$) are present. There are different equivalent forms of this equation used in the literature.[6–10] The compressible flow Poisson equation for $p'$ is also the starting point for obtaining the Lighthill acoustic analogy equation for the acoustic pressure,[11] *ie* the part $p'_{\mathrm{a}}$ of $p'$ which will be propagated to the farfield as sound, under the assumption $dp'_{\mathrm{a}} \cong a^2 d\rho'$.

The incompressible flow Poisson equation (Eq. 1) introduces the idea[5] that solenoidal[12] pressure fluctuations, associated with the fluctuating velocity field, are generated by 2 separate mechanisms:[4,13–17] 1) by the interaction of velocity fluctuations with mean-velocity-gradients (termed rapid pressure fluctuations, because they interact immediately with an imposed mean-velocity-gradient, or linear pressure fluctuations, because the corresponding source-term is linear in velocity fluctuations), and 2) by the turbulence-turbulence interaction (termed slow pressure fluctuations, because they react much slower than the rapid ones which are directly driven by mean-velocity-gradients, or nonlinear pressure fluctuations, because the corresponding source-term is quadratic in velocity fluctuations). This idea of distinguishing between pressure fluctuations associated with mean-flow-gradients ($p'_{(\mathrm{r})}$) and pressure fluctuations associated with turbulence-turbulence interactions only ($p'_{(\mathrm{s})}$) is applied in general for all correlations which contain the fluctuating pressure.[4,13–17]

Chou[5] pointed out that, because of the linearity in $p'$ of the incompressible flow Poisson equation (Eq. 1) separate solutions can be obtained for each of the 2 source-terms, based on the free-space Green's function

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for the Poisson equation[18]

$$[M \longrightarrow 0\ ;\ \rho,\ \mu \cong \text{const}]:$$

$$\frac{1}{\bar{\rho}} p'_{(\mathrm{r})}(\vec{x},t) = \underbrace{\frac{1}{2\pi}\iiint_{\mathfrak{V}} \left(\frac{\partial \overline{\mathfrak{u}}_\ell}{\partial \mathfrak{x}_k}\frac{\partial \mathfrak{u}'_k}{\partial \mathfrak{x}_\ell}\right)\frac{d\mathfrak{v}(\vec{\mathfrak{x}})}{|\vec{\mathfrak{x}}-\vec{x}|}}_{p'_{(\mathrm{r};\mathfrak{V})}(\vec{x},t)} + \underbrace{\frac{1}{4\pi\bar{\rho}}\iint_{\partial\mathfrak{V}}\left(\frac{1}{|\vec{\mathfrak{x}}-\vec{x}|}\frac{\partial \mathfrak{p}'_{\mathrm{r}}}{\partial \mathfrak{n}} - \mathfrak{p}'_{\mathrm{r}}\frac{\partial}{\partial \mathfrak{n}}\left[\frac{1}{|\vec{\mathfrak{x}}-\vec{x}|}\right]\right) d\mathfrak{S}}_{p'_{(\mathrm{r};w)}(\vec{x},t)} \tag{2}$$

$$\frac{1}{\bar{\rho}} p'_{(\mathrm{s})}(\vec{x},t) = \underbrace{\frac{1}{4\pi}\iiint_{\mathfrak{V}} \left(\frac{\partial^2 \mathfrak{u}'_k \mathfrak{u}'_\ell}{\partial \mathfrak{x}_\ell \partial \mathfrak{x}_k} - \frac{\partial^2 \overline{\mathfrak{u}'_k \mathfrak{u}'_\ell}}{\partial \mathfrak{x}_\ell \partial \mathfrak{x}_k}\right)\frac{d\mathfrak{v}(\vec{\mathfrak{x}})}{|\vec{\mathfrak{x}}-\vec{x}|}}_{p'_{(\mathrm{s};\mathfrak{V})}(\vec{x},t)} + \underbrace{\frac{1}{4\pi\bar{\rho}}\iint_{\partial\mathfrak{V}}\left(\frac{1}{|\vec{\mathfrak{x}}-\vec{x}|}\frac{\partial \mathfrak{p}'_{\mathrm{s}}}{\partial \mathfrak{n}} - \mathfrak{p}'_{\mathrm{s}}\frac{\partial}{\partial \mathfrak{n}}\left[\frac{1}{|\vec{\mathfrak{x}}-\vec{x}|}\right]\right) d\mathfrak{S}}_{p'_{(\mathrm{s};w)}(\vec{x},t)} \tag{3}$$

where $p$, $u$ are the pressure and velocity at point $\vec{x}$, and the volume and surface integrals are taken over all other points $\vec{\mathfrak{x}}$ where the pressure and velocity are $\mathfrak{p}$ and $\mathfrak{u}$. Notice that if (Eqs. 2, 3) are multiplied by a function of $\vec{x}$, this function can be entered into the integrals which are over $\vec{\mathfrak{x}}$. In the case of unbounded flow, where $\partial\mathfrak{V}$ is very far (at infinity), only the volume integrals remain. On the other hand, for flow near solid boundaries, the surface integrals indicate that the unsteady pressure field reacts to the presence of the wall (surface integral; $\mathfrak{n}$ is the normal to the wall, directed outrwards). Terms related to the surface integrals are usually called wall-echo-terms (also known as wall blockage effect[19]), since for an infinite plane solid boundary they can be related to reflection from the wall (method of images[4,16,17]).

Starting from the seminal paper of Chou,[5] all models[4,13–17] for the redistribution tensor $\phi_{ij}$, the velocity/pressure-gradient tensor $\Pi_{ij}$ or the pressure transport $\overline{p'u'_i}$ appearing in the pressure-diffusion tensor $d^{(p)}_{ij}$, are traditionally composed of 4 parts corresponding to the splitting $p' = p'_{(\mathrm{r};\mathfrak{V})} + p'_{(\mathrm{r};w)} + p'_{(\mathrm{s};\mathfrak{V})} + p'_{(\mathrm{s};w)}$ (Eqs. 1–3). There is at present no possibility to separately measure $p'_{(\mathrm{r};\mathfrak{V})}$, $p'_{(\mathrm{r};w)}$, $p'_{(\mathrm{s};\mathfrak{V})}$, and $p'_{(\mathrm{s};w)}$, and even the simulatneous measurement of $p'$ and $\partial u'_i/\partial x_j$ in the flowfield seems beyond the present measurement technology state-of-the-art.[?] Direct numerical simulation (DNS) offers the possibility to directly compute the different terms.

Kim,[20] in the context of incompressible flow pseudospectral plane-channel DNS,[21,22] used a Green's function approach[23–28] to solve, as a function of $y$ (normal-to-the-wall coordianate), the incompressible flow Poisson equation for $p'$, for each parallel-to-the-wall wavenumber $\kappa_x$ and $\kappa_z$. The separate solution for each source-term, permits the separation of slow $p'_{(\mathrm{s})}$ and rapid $p'_{(\mathrm{r})}$ parts. In this way,[20] the slow and rapid contributions from different location in the flowfield to $\phi_{ij}$ were computed. This procedure[20] is now used in a standard way in incompressible plane-channel flow DNS, at least as far as $\overline{p'^2_{(\mathrm{s})}}$, $\overline{p'^2_{(\mathrm{r})}}$, and $\overline{p'^2_{(\tau)}}$ are concerned ($\overline{p'^2_{(\tau)}}$, usually called Stokes pressure,[29] corresponds to the separately computed contribution of the wall-boundary condition[4] $[\partial_n p']_w \cong [\mu\partial^2_{nn}u_n]_w\ ;\ [M \longrightarrow 0; \rho,\ \mu \cong \text{const}]$, $n$ being the normal-to-the-wall direction; it is stressed, in the preset work that this boundary-condition is associated with the source-term $\partial^2_{x_i x_j}[\tau'_{ij}]$ in the Poisson equation for $p'$, which is equal to 0 at the incompressible flow limit).[21,22,30,31] This procedure was later used by Chang et al.[29] to study the detailed space-frequency contributions of the flowfield to wall-pressure wavenumber-frequency spectra.

DNS results for $\phi_{ij}$, $\Pi_{ij}$ and $\overline{p'u'_i}$ are systematically used for the *a priori* and *a posteriori* evaluation of modelling proposals.[32,33] Nonetheless, to the authors' knowledge, the detailed 4-part decomposition used in the models has not been studied in detail, nor have DNS data been used to further split $p'_{(\mathrm{s})}$ and $p'_{(\mathrm{r})}$ to volume ($p'_{(\mathrm{r};\mathfrak{V})}$ and $p'_{(\mathrm{s};\mathfrak{V})}$) and wall-echo ($p'_{(\mathrm{r};w)}$ and $p'_{(\mathrm{s};w)}$) terms.

The purpose of the present paper is to revisit the processing of DNS computations to obtain the 5-part decomposition of various $p'$-related correlations appearing in the Reynolds-stress transport equations. The DNS code used[34,35] solves the compressible Navier-Stokes equations, and in this way computes $p'$ directly, and does not require the solution of a Poisson equation for pressure, as part of the algorithm. This directly computed $p'$ is used to check that the sum of the 5-part-split terms is correct. A processing module is developed, using a Green's function solution for ($p'_{(\mathrm{r})}$, $p'_{(\mathrm{s})}$ and $p'_{(\tau)}$),[20] augmented with free-space Green-function solutions ($p'_{(\mathrm{r};\mathfrak{V})}$), $p'_{(\mathrm{r};\mathfrak{V})}$) and $p'_{(\tau)}$). The difference between these 2 solutions allows the evaluation of the wall-echo contribution to the pressure-fluctuations. The contribution of each of the 4 components of $p'$ to $\phi_{ij}$, $\Pi_{ij}$ and $\overline{p'u'_i}$ is computed. These results can be used as a database for the analysis and improvement of closures for these terms in the context of Reynolds-stress models (RSMs).[36]

## II. Reynolds-Stress Transport and Modelling

The flow is modelled by the compressible Navier-Stokes equations, with a volume-force $\rho f_{\mathrm{V}_i}$. In the general case the body-acceleration $f_{\mathrm{V}_i}$ may include gravity, rotating frame-of-reference centrifugal and Coriolis accelerations, *etc*. In the present DNS computations, following Coleman et al.,[37,38] a spatially constant $f_{\mathrm{V}_x}$ is adjusted to obtain fully developed compressible channel flow[39] (*cf* §IV.A)

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_\ell}[\rho u_\ell] = 0 \tag{4}$$

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial}{\partial x_\ell}[\rho u_i u_\ell] = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{i\ell}}{\partial x_\ell} + \rho f_{\mathrm{V}_i} \tag{5}$$

$$\frac{\partial}{\partial t}[\rho h_t - p] + \frac{\partial}{\partial x_\ell}[\rho h_t u_\ell] = \frac{\partial}{\partial x_\ell}[u_i \tau_{i\ell} - q_\ell] + \rho f_{\mathrm{V}_i} u_i \tag{6}$$

where $t$ is the time, $x_i$ are the cartesian space coordinates, $u_i$ are the velocity components, $\rho$ is the density, $p$ is the pressure, $h$ is the enthalpy, $h_t = h + \frac{1}{2}u_i u_i$ is the total enthalpy, $\tau_{ij}$ is the viscous-stress-tensor, and $q_i$ is the molecular heat-flux.

Introducing Reynolds-averages,[4] *eg*

$$\rho = \bar{\rho} + \rho' \qquad ; \qquad \overline{\rho'} = 0 \tag{7}$$

and Favre-averages,[40] *eg*

$$u_i = \bar{u}_i + u_i' = \tilde{u}_i + u_i'' \qquad ; \qquad \overline{\rho u_i''} = 0 \qquad ; \qquad \overline{\rho u_i} = \bar{\rho}\tilde{u}_i \qquad ; \qquad \overline{u_i''} = -\frac{\overline{\rho' u_i''}}{\bar{\rho}} = -\frac{\overline{\rho' u_i'}}{\bar{\rho}} \tag{8}$$

the Reynolds-averaged Navier-Stokes equations read[40]

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x_\ell}[\bar{\rho}\tilde{u}_\ell] = 0 \tag{9}$$

$$\frac{\partial \bar{\rho}\tilde{u}_i}{\partial t} + \frac{\partial}{\partial x_\ell}[\bar{\rho}\tilde{u}_i\tilde{u}_\ell] = -\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial \bar{\tau}_{i\ell}}{\partial x_\ell} - \frac{\partial \bar{\rho}\widetilde{u_i'' u_\ell''}}{\partial x_\ell} + \overline{\rho f_{\mathrm{V}_i}} \tag{10}$$

$$\frac{\partial}{\partial t}[\bar{\rho}\breve{h}_t - \bar{p}] + \frac{\partial}{\partial x_\ell}[\bar{\rho}\breve{h}_t\tilde{u}_\ell] = \frac{\partial}{\partial x_\ell}[\tilde{u}_i(\bar{\tau}_{i\ell} - \bar{\rho}\widetilde{u_i'' u_\ell''}) - (\bar{q}_\ell + \bar{\rho}\widetilde{h'' u_\ell''})] + \overline{\rho f_{\mathrm{V}_i} u_i} + S_{\breve{h}_t} \tag{11}$$

where $\breve{h}_t = \tilde{h}_t + \frac{1}{2}\tilde{u}_i\tilde{u}_i$ is the meanflow-total-enthalpy[41,42] or computable total enthalpy, and the source-term in the Reynolds-averaged energy equation is

$$S_{\breve{h}_t} = -(P_{\mathrm{k}} - \bar{\rho}\varepsilon + \overline{p'\frac{\partial u_\ell''}{\partial x_\ell}}) + \frac{\partial}{\partial x_\ell}[\overline{p u_\ell''}] + (-\bar{p}\delta_{i\ell} + \bar{\tau}_{i\ell})\frac{\partial \overline{u_i''}}{\partial x_\ell} = -(P_{\mathrm{k}} - \bar{\rho}\varepsilon) + \overline{u_\ell''\frac{\partial p}{\partial x_\ell}} + \bar{\tau}_{i\ell}\frac{\partial \overline{u_i''}}{\partial x_\ell} \tag{12}$$

where $P_{\mathrm{k}} = \frac{1}{2}P_{\ell\ell} = -\bar{\rho}\widetilde{u_i'' u_j''}\partial_{x_j}\tilde{u}_i$ is the production of turbulence-kinetic-energy, and $\varepsilon$ is the turbulence-kinetic-energy dissipation-rate. The exact transport equations for the Favre-Reynolds-averaged Reynolds-stresses are[43,44]

$$\begin{aligned}
\frac{\partial \bar{\rho}\widetilde{u_i'' u_j''}}{\partial t} + \frac{\partial(\bar{\rho}\widetilde{u_i'' u_j''}\tilde{u}_\ell)}{\partial x_\ell} &= \frac{\partial}{\partial x_\ell}(-\bar{\rho}\widetilde{u_i'' u_j'' u_\ell''} - \overline{p' u_j''}\delta_{i\ell} - \overline{p' u_i''}\delta_{j\ell} + \overline{u_i''\tau_{j\ell}'} + \overline{u_j''\tau_{i\ell}'}) + \left(-\bar{\rho}\widetilde{u_i'' u_\ell''}\frac{\partial \tilde{u}_j}{\partial x_\ell} - \bar{\rho}\widetilde{u_j'' u_\ell''}\frac{\partial \tilde{u}_i}{\partial x_\ell}\right) \\
&\quad + \overline{p'\left(\frac{\partial u_i''}{\partial x_j} + \frac{\partial u_j''}{\partial x_i} - \frac{2}{3}\frac{\partial u_k''}{\partial x_k}\delta_{ij}\right)} + \frac{2}{3}\overline{p'\frac{\partial u_k''}{\partial x_k}}\delta_{ij} - \left(\overline{\tau_{j\ell}'\frac{\partial u_i''}{\partial x_\ell}} + \overline{\tau_{i\ell}'\frac{\partial u_j''}{\partial x_\ell}}\right) \\
&\quad + \left(-\overline{u_i''}\frac{\partial \bar{p}}{\partial x_j} - \overline{u_j''}\frac{\partial \bar{p}}{\partial x_i} + \overline{u_i''}\frac{\partial \bar{\tau}_{j\ell}}{\partial x_\ell} + \overline{u_j''}\frac{\partial \bar{\tau}_{i\ell}}{\partial x_\ell}\right) + \left(\overline{\rho u_j'' f_{\mathrm{V}_i}'' + \rho u_i'' f_{\mathrm{V}_j}''}\right)
\end{aligned} \tag{13}$$

It is known that the averages of mixed Reynolds- and Favre-fluctuations, for any 2 random variables $\mathrm{a}$ and $b$ satisfies[40]

$$\overline{\mathrm{a}''\mathrm{b}'} = \overline{\mathrm{a}'\mathrm{b}''} = \overline{\mathrm{a}'\mathrm{b}'} \tag{14}$$

because (Eqs. 7, 8) $\overline{\mathrm{a}''\mathrm{b}'} = \overline{(\mathrm{a}' + \bar{\mathrm{a}} - \tilde{\mathrm{a}})\mathrm{b}'} = \overline{\mathrm{a}'\mathrm{b}'} + (\bar{\mathrm{a}} - \tilde{\mathrm{a}})\overline{\mathrm{b}'} = \overline{\mathrm{a}'\mathrm{b}'}$. This property (Eq. 14) was used in the last relation of (Eqs. 8). Notice that the same is true for

$$\overline{\mathrm{a}'\frac{\partial \mathrm{b}''}{\partial x_\ell}} = \overline{\mathrm{a}'\frac{\partial(\mathrm{b}' + \bar{\mathrm{b}} - \tilde{\mathrm{b}})}{\partial x_\ell}} = \overline{\mathrm{a}'\frac{\partial \mathrm{b}'}{\partial x_\ell}} + \overline{\mathrm{a}'}\overline{\frac{\partial(\bar{\mathrm{b}} - \tilde{\mathrm{b}})}{\partial x_\ell}} = \overline{\mathrm{a}'\frac{\partial \mathrm{b}'}{\partial x_\ell}} \tag{15}$$

This property is not trivial because Favre-averaging does not commute with differentiation ($\widetilde{\partial_{x_j} u_i} \neq \partial_{x_j} \tilde{u}_i$). Finally, for any 2 random variables $\mathrm{a}$ and $\mathrm{b}$, from the definition of Favre-averaging (Eq. 8), $\overline{\rho \mathrm{a}''\mathrm{b}''} = \overline{\rho \mathrm{a}''(\mathrm{b}' + \bar{\mathrm{b}} - \tilde{\mathrm{b}})} = \overline{\rho \mathrm{a}''\mathrm{b}'} + (\bar{\mathrm{b}} - \tilde{\mathrm{b}})\overline{\rho \mathrm{a}''} = \overline{\rho \mathrm{a}''\mathrm{b}'}$, so that

$$\overline{\rho \mathrm{a}''\mathrm{b}''} = \overline{\rho \mathrm{a}''\mathrm{b}'} = \overline{\rho \mathrm{a}'\mathrm{b}''} \tag{16}$$

Using these properties (Eqs. 14–16) the exact transport equations for the Reynolds-stresses (Eq. 13) read

$$\begin{aligned}
\underbrace{\frac{\partial \bar{\rho}\widetilde{u_i'' u_j''}}{\partial t} + \frac{\partial(\bar{\rho}\widetilde{u_i'' u_j''}\tilde{u}_\ell)}{\partial x_\ell}}_{\text{convection } C_{ij}} &= \underbrace{\frac{\partial}{\partial x_\ell}(-\bar{\rho}\widetilde{u_i'' u_j'' u_\ell''} - \overline{p' u_j'}\delta_{i\ell} - \overline{p' u_i'}\delta_{j\ell} + \overline{u_i' \tau_{j\ell}'} + \overline{u_j' \tau_{i\ell}'})}_{\text{diffusion } d_{ij} = d_{ij}^{(u)} + d_{ij}^{(p)} + d_{ij}^{(\mu)}} + \underbrace{\left(-\bar{\rho}\widetilde{u_i'' u_\ell''}\frac{\partial \tilde{u}_j}{\partial x_\ell} - \bar{\rho}\widetilde{u_j'' u_\ell''}\frac{\partial \tilde{u}_i}{\partial x_\ell}\right)}_{\text{production } P_{ij}} \\
&+ \underbrace{\overline{p'\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i} - \tfrac{2}{3}\frac{\partial u_k'}{\partial x_k}\delta_{ij}\right)}}_{\text{redistribution } \phi_{ij}} + \underbrace{\tfrac{2}{3}\overline{p'\frac{\partial u_k'}{\partial x_k}}\delta_{ij}}_{\text{pressure-dilatation } \phi_p\delta_{ij}} - \underbrace{\left(\overline{\tau_{j\ell}'\frac{\partial u_i'}{\partial x_\ell}} + \overline{\tau_{i\ell}'\frac{\partial u_j'}{\partial x_\ell}}\right)}_{\text{dissipation } \bar{\rho}\varepsilon_{ij}} \\
&+ \underbrace{\left(-\overline{u_i''}\frac{\partial \bar{p}}{\partial x_j} - \overline{u_j''}\frac{\partial \bar{p}}{\partial x_i} + \overline{u_i''}\frac{\partial \bar{\tau}_{j\ell}}{\partial x_\ell} + \overline{u_j''}\frac{\partial \bar{\tau}_{i\ell}}{\partial x_\ell}\right)}_{\text{density fluctuation effects } K_{ij}} + \underbrace{\left(\overline{\rho u_j'' f_{\mathrm{V}_i}' + \rho u_i'' f_{\mathrm{V}_j}'}\right)}_{\text{bodyforce fluctuation effects } B_{ij}}
\end{aligned} \tag{17}$$

where diffusion $d_{ij}$ is split into 3 parts, pressure-diffusion $d_{ij}^{(p)} := -\partial_{x_\ell}(\overline{p' u_j'}\delta_{i\ell} + \overline{p' u_i'}\delta_{j\ell})$, diffusion by triple-velocity-correlations $d_{ij}^{(p)} := -\partial_{x_\ell}(\bar{\rho}\widetilde{u_i'' u_j'' u_\ell''})$, and to viscous diffusion $d_{ij}^{(\mu)} := \partial_{x_\ell}(\overline{u_i' \tau_{j\ell}'} + \overline{u_j' \tau_{i\ell}'})$. Notice that the terms $\phi_{ij}$, $\phi_p$, and $d_{ij}^{(p)}$ can be recombined into a single term

$$\Pi_{ij} := \overline{u_i' \frac{\partial p'}{\partial x_j}} + \overline{u_j' \frac{\partial p'}{\partial x_i}} \equiv \phi_{ij} + \phi_p \delta_{ij} + d_{ij}^{(p)} \tag{18}$$

The velocity/pressure-gradient tensor $\Pi_{ij}$ was used in the original formulation by Chou,[5] but also more recently by several authors.

Modelling the terms containing the pressure-fluctuation $p'$, *ie* redistribution $\phi_{ij}$, pressure diffusion $d_{ij}^{(p)}$, and pressure-dilatation $\phi_p$, requires the analysis of the dynamics of $p'$. This, following Chou,[5] is traditionally based on the incompressible-flow Poisson equation for $p'$, although, several authors,[6–10] have contributed towards extending this approach to the compressible flow Poisson equation for $p'$. Notice that because of the mixed-fluctuations-product-averaging properties (Eqs. 14, 14), these terms ($\phi_{ij}$, $d_{ij}^{(p)}$, $\phi_p$) are written as averages of nonweighted (Reynolds) fluctuations (Eqs. 17).

## III. Pressure-Fluctuations in Compressible Flow

### A. The Poisson Equation for $p$

The Poisson equation for $p$, in compressible flow is derived taking the divergence of the momentum equations (Eq. 5)

$$\frac{\partial^2}{\partial x_i \partial t}(\rho u_i) + \frac{\partial^2}{\partial x_i \partial x_j}(\rho u_i u_j) = -\nabla^2 p + \frac{\partial^2 \tau_{ij}}{\partial x_i \partial x_j} + \frac{\partial}{\partial x_i}(\rho f_{\mathrm{V}_i}) \tag{19}$$

Derivation of the continuity equations (Eq. 4) respectively by $\partial_t$, $\partial_{x_i}$, and $D_t$, yields the relations

$$\frac{\partial^2}{\partial x_i \partial t}(\rho u_i) = -\frac{\partial^2 \rho}{\partial t^2} \tag{20}$$

$$\frac{\partial^2}{\partial x_i \partial x_j}(\rho u_i) = -\frac{\partial^2 \rho}{\partial x_j \partial t} \tag{21}$$

$$\left[\frac{D_c}{Dt}\right]^2(\rho) := \frac{\partial^2 \rho}{\partial t^2} + 2u_j \frac{\partial^2 \rho}{\partial x_j \partial t} + u_i u_j \frac{\partial^2 \rho}{\partial x_i \partial x_j} = -\frac{D}{Dt}(\rho\Theta) - \frac{D\vec{V}}{Dt} \cdot \mathrm{grad}\rho \tag{22}$$

where $\Theta := S_{\ell\ell} = \partial_{x_\ell} u_\ell$ is the dilatation, $D_t := \partial_t + u_i \partial_{x_i}$ is the substantial or material derivarive,[4] and the operator $[D_{c_t}]^2$ is the Garrick operator,[45] which is used in theoretical compressible unsteady aerodynamics and aeroacoustics[46,47]

$$\left[\frac{D}{Dt}\right]^2 (\rho) \; = \left[\frac{D_c}{Dt}\right]^2 (\rho) \; + \frac{D\vec{V}}{Dt} \cdot \mathrm{grad}\rho \tag{23}$$

and corresponds to treating the velocity $u_i$ entering in the definition of the substantial derivative as constant when computing $[D_{c_t}]^2 \equiv D_{c_t}[D_{c_t}]$.[46,47]

Replacing the above relations (Eqs. 20–22) into (Eq. 19) yields 3 alternative expressions for $\nabla^2 p$

$$\nabla^2 p = \frac{\partial^2 \tau_{ij}}{\partial x_i \partial x_j} + \frac{\partial}{\partial x_i}(\rho f_{\mathrm{V}_i}) - \frac{\partial^2}{\partial x_i \partial x_j}(\rho u_i u_j) + \frac{\partial^2 \rho}{\partial t^2} \tag{24}$$

$$\nabla^2 p = \frac{\partial^2 \tau_{ij}}{\partial x_i \partial x_j} + \frac{\partial}{\partial x_i}(\rho f_{\mathrm{V}_i}) - \rho \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i} - \rho\Theta^2 + \left[\frac{D_c}{Dt}\right]^2 (\rho) \tag{25}$$

$$\nabla^2 p = \frac{\partial^2 \tau_{ij}}{\partial x_i \partial x_j} + \frac{\partial}{\partial x_i}(\rho f_{\mathrm{V}_i}) - \rho \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i} - \rho\frac{D\Theta}{Dt} - \frac{D\vec{V}}{Dt} \cdot \mathrm{grad}\rho \tag{26}$$

differing in the way the terms coming from taking the divergence of convection are expressed. The first expression (Eq. 24), which is the most concise, and has been used, *eg* by Strahle,[6] Taulbee and VanOsdol,[7] Lee et al.,[8] and Adumitroaie et al.,[9] in the context of compressible RSM closure development. However the separation of what are actually convective compressible effects (related to the $\Theta$ or to $[D_{c_t}]^2\rho$ form) are probably better separated from the incompressible-like term $\rho[\partial_{x_j} u_i][\partial_{x_i} u_j]$ in the second expression (Eq. 25) which was used by Foysi et al..[10] The third expression (Eq. 26), which was used in the present work expresses these convective purely compressible effects in a form containing an acceleration/density-gradient interaction term ($[D_t u_i][\partial_{x_i}\rho]$; Eq. 26) A comparative study between these 3 forms (Eqs. 24–26) is necessary to highlight the physical significance of the various terms, and to provide guidance on modelling practices. This will be the subject of future work.

The boundary conditions at a fixed solid wall $\partial\mathfrak{V}_w$ ($u_i(t,\vec{x}) = 0 \;\forall\; \vec{x} \in \partial\mathfrak{V}_w$) are determined by projecting the momentum equation (Eq. 5), written in nonconservative form, on the normal-to-the-wall direction $\vec{e}_n$

$$\frac{\partial p}{\partial n} = \frac{\partial \tau_{nj}}{\partial x_j} + \rho f_{\mathrm{V}_n} \quad \forall\; \vec{x} \in \partial\mathfrak{V}_w \tag{27}$$

$n$ being the coordinate along $\vec{e}_n$.

## B. The Poisson Equation for $p'$

Computing the fluctuating part of the last form of the Poisson equation for $p$ (Eq. 26) yields

$$\begin{aligned}
\nabla^2 p' = {} & \underbrace{\frac{\partial^2 \tau_{ij'}}{\partial x_i \partial x_j}}_{Q'_{(\tau)}} + \underbrace{\frac{\partial}{\partial x_i}(\rho f_{\mathrm{V}_i} - \overline{\rho f_{\mathrm{V}_i}})}_{Q'_{(\mathrm{BF})}} \\
& + \underbrace{\left[-\bar{\rho}\left(\frac{\partial u_i'}{\partial x_j}\frac{\partial u_j'}{\partial x_i} - \overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_j'}{\partial x_i}}\right)\right]}_{Q'_{(\mathrm{s})}} + \underbrace{\left[-\left(\rho'\frac{\partial u_i'}{\partial x_j}\frac{\partial u_j'}{\partial x_i} - \overline{\rho'\frac{\partial u_i'}{\partial x_j}\frac{\partial u_j'}{\partial x_i}}\right)\right]}_{Q'_{(\rho'\mathrm{s})}} \\
& + \underbrace{\left[-2\bar{\rho}\left(\frac{\partial u_i'}{\partial x_j}\frac{\partial \bar{u}_j}{\partial x_i}\right)\right]}_{Q'_{(\mathrm{r})}} + \underbrace{\left[-2\left(\rho'\frac{\partial u_i'}{\partial x_j} - \overline{\rho'\frac{\partial u_i'}{\partial x_j}}\right)\frac{\partial \bar{u}_j}{\partial x_i}\right]}_{Q'_{(\rho'\mathrm{r})}} + \underbrace{\left[-\rho'\frac{\partial \bar{u}_i}{\partial x_j}\frac{\partial \bar{u}_j}{\partial x_i}\right]}_{Q'_{(\rho')}} \\
& + \underbrace{\left[-\left(\rho'\overline{\frac{D\Theta}{Dt}} + \rho\left[\frac{D\Theta}{Dt}\right]' - \overline{\rho'\frac{D\Theta'}{Dt}}\right)\right]}_{Q'_{(\Theta)}} + \underbrace{\left[-\left(\left[\frac{Du_i}{Dt}\right]'\frac{\partial \bar{\rho}}{\partial x_i}\right)\right]}_{Q'_{(\dot{V}\nabla\bar{\rho})}} + \underbrace{\left[-\left(\frac{Du_i}{Dt}\frac{\partial \rho'}{\partial x_i} - \overline{\left[\frac{Du_i}{Dt}\right]'\frac{\partial \rho'}{\partial x_i}}\right)\right]}_{Q'_{(\dot{V}\nabla\rho')}}
\end{aligned} \tag{28}$$

where

$$\overline{\frac{D\Theta}{Dt}} := \overline{\frac{\partial\Theta}{\partial t} + u_j\frac{\partial\Theta}{\partial x_j}} = \frac{\partial\bar{\Theta}}{\partial t} + \bar{u}_j\frac{\partial\bar{\Theta}}{\partial x_j} + \overline{u'_j\frac{\partial\Theta'}{\partial x_j}} \tag{29}$$

$$\left[\frac{D\Theta}{Dt}\right] := \frac{D\Theta}{Dt} - \overline{\frac{D\Theta}{Dt}} = \frac{\partial\Theta'}{\partial t} + \bar{u}_j\frac{\partial\Theta'}{\partial x_j} + u'_j\frac{\partial\bar{\Theta}}{\partial x_j} + u'_j\frac{\partial\Theta'}{\partial x_j} - \overline{u'_j\frac{\partial\Theta'}{\partial x_j}} \tag{30}$$

with similar relations for $\overline{D_t u_i}$ and $[D_t u_i]'$. The Neumann boundary-conditions[27] of the Poisson equation for $p'$ (Eq. 28), at a solid wall, are readily obtained by considering the fluctuating part of (Eq. 31)

$$\frac{\partial p'}{\partial n} = \frac{\partial\tau'_{nj}}{\partial x_j} + [\rho f_{\mathrm{V}_n}]' \quad \forall\, \vec{x} \in \partial\mathfrak{V}_w \tag{31}$$

## C. $p'$ Splitting

Because of the linearity of the $\nabla^2$ Laplacian operator, each term in (Eq. 28) can be treated separately

$$\nabla^2 p'_\tau = Q'_{(\tau)} \quad ; \quad \frac{\partial p'_{(\tau)}}{\partial n} = \frac{\partial\tau'_{nj}}{\partial x_j} \quad \forall\vec{x} \in \partial\mathfrak{V}_w \tag{32}$$

$$\nabla^2 p'_{\mathrm{BF}} = Q'_{(\mathrm{BF})} \quad ; \quad \frac{\partial p'_{(\mathrm{BF})}}{\partial n} = [\rho f_{\mathrm{V}_n}]' \quad \forall\vec{x} \in \partial\mathfrak{V}_w \tag{33}$$

$$\nabla^2 p'_s = Q'_{(\mathrm{s})} \quad ; \quad \frac{\partial p'_{(\mathrm{s})}}{\partial n} = 0 \quad \forall\vec{x} \in \partial\mathfrak{V}_w \tag{34}$$

$$\nabla^2 p'_{(\rho' s)} = Q'_{(\rho'\mathrm{s})} \quad ; \quad \frac{\partial p'_{(\rho'\mathrm{s})}}{\partial n} = 0 \quad \forall\vec{x} \in \partial\mathfrak{V}_w \tag{35}$$

$$\nabla^2 p'_r = Q'_{(\mathrm{r})} \quad ; \quad \frac{\partial p'_{(\mathrm{r})}}{\partial n} = 0 \quad \forall\vec{x} \in \partial\mathfrak{V}_w \tag{36}$$

$$\nabla^2 p'_{(\rho' r)} = Q'_{(\rho'\mathrm{r})} \quad ; \quad \frac{\partial p'_{(\rho'\mathrm{r})}}{\partial n} = 0 \quad \forall\vec{x} \in \partial\mathfrak{V}_w \tag{37}$$

$$\nabla^2 p'_{\rho'} = Q'_{(\rho')} \quad ; \quad \frac{\partial p'_{(\rho')}}{\partial n} = 0 \quad \forall\vec{x} \in \partial\mathfrak{V}_w \tag{38}$$

$$\nabla^2 p'_\Theta = Q'_{(\Theta)} \quad ; \quad \frac{\partial p'_{(\Theta)}}{\partial n} = 0 \quad \forall\vec{x} \in \partial\mathfrak{V}_w \tag{39}$$

$$\nabla^2 p'_{(\dot{V}\nabla\bar{\rho})} = Q'_{(\dot{V}\nabla\bar{\rho})} \quad ; \quad \frac{\partial p'_{(\dot{V}\nabla\bar{\rho})}}{\partial n} = 0 \quad \forall\vec{x} \in \partial\mathfrak{V}_w \tag{40}$$

$$\nabla^2 p'_{(\dot{V}\nabla\rho')} = Q'_{(\dot{V}\nabla\rho')} \quad ; \quad \frac{\partial p'_{(\dot{V}\nabla\rho')}}{\partial n} = 0 \quad \forall\vec{x} \in \partial\mathfrak{V}_w \tag{41}$$

where, obviously, the boundary-conditions (Eq. 31) are associated with $p'_\tau$ (Eq. 43) and $p'_{\mathrm{BF}}$ (Eq. 44). Notice the $[\rho f_{\mathrm{V}_n}]'$ is identically equal to 0 if $\vec{f}_{\mathrm{V}} \perp \vec{e}_n$. Each of the 10 source-terms (Eqs. 43–41) has a Reynolds-average equal to 0 (each term is a fluctuation).

## D. Limiting Case of Incompressible Constant Viscosity Flow

In the limiting case of incompressible ($\rho \cong \mathrm{const}$ so that $\bar{\rho} \cong \mathrm{const}$ and $\rho' \cong 0$) constant viscosity flow ($\mu \cong \mathrm{const}$) it is easy to show that

$$[M \longrightarrow 0\ ;\ \rho,\ \mu \cong \mathrm{const}] \implies Q_{(\tau)} \cong Q'_{(\rho'\mathrm{s})} \cong Q'_{(\rho'\mathrm{r})} \cong Q'_{(\rho')} \cong Q'_{(\Theta)} \cong Q'_{(\dot{V}\nabla\bar{\rho})} \cong Q'_{(\dot{V}\nabla\rho')} \cong 0 \tag{42}$$

and pressure is split to only 4 terms ($p' = p'_{(\tau)} + p'_{(\mathrm{BF})} + p'_{(\mathrm{s})} + p'_{(\mathrm{r})}$)

$[M \longrightarrow 0\ ;\ \rho,\ \mu \cong \mathrm{const}] \implies$

$$\nabla^2 p'_{(\tau)} \cong 0 \quad ; \quad \frac{\partial p'_{(\tau)}}{\partial n} = \frac{\partial \tau'_{nj}}{\partial x_j} \cong \bar{\mu}\frac{\partial^2 u'_n}{\partial n^2} \quad \forall \vec{x} \in \partial\mathfrak{V}_w \tag{43}$$

$$\nabla^2 p'_{(\mathrm{BF})} = Q'_{(\mathrm{BF})} \quad ; \quad \frac{\partial p'_{(\mathrm{BF})}}{\partial n} = [\rho f_{\mathrm{V}_n}]' \quad \forall \vec{x} \in \partial\mathfrak{V}_w \tag{44}$$

$$\nabla^2 p'_{(\mathrm{s})} = Q'_{(\mathrm{s})} \quad ; \quad \frac{\partial p'_{(\mathrm{s})}}{\partial n} = 0 \quad \forall \vec{x} \in \partial\mathfrak{V}_w \tag{45}$$

$$\nabla^2 p'_{(r)} = Q'_{(\mathrm{r})} \quad ; \quad \frac{\partial p'_{(\mathrm{r})}}{\partial n} = 0 \quad \forall \vec{x} \in \partial\mathfrak{V}_w \tag{46}$$

The interesting point in this limiting procedure, is that it gives a clearer physical significance to the so-called boundary-condition term,[10,29] $p'_{(\tau)}$.

# IV. DNS Computations

## A. Flow Model

The flow is modelled by the compressible Navier-Stokes equations, with a volume-force $\rho f_{\mathrm{V}_x}$ in the streamwise direction (the acceleration $f_{\mathrm{V}_x}$ being spatially constant), which, following Coleman et al.,[37,38] is adjusted to counteract viscous friction, so that a fully established compressible channel flow ($\partial\bar{p}/\partial x = 0$, $\partial\bar{\rho}/\partial x = 0$, $\partial\bar{u}_i/\partial x = 0$)

Perfect gas thermodynamics with constant $c_p$ were assumed[48]

$$p = \rho R_g T \quad ; \quad R_g = \mathrm{const} \quad ; \quad c_p = \frac{\gamma}{\gamma - 1} R_g = \mathrm{const} \tag{47}$$

where $T$ is the temperature, $R_g$ is the gas-constant, $\gamma$ is the isentropic exponent and $c_p$ is the heat-capacity at constant pressure. Standard linear laws were used for the constitutive relation for the stress-tensor and the heat-flux vector[49]

$$\tau_{ij} = \mu(T)\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \tfrac{2}{3}\frac{\partial u_\ell}{\partial x_\ell}\delta_{ij}\right] + \mu_{\mathrm{b}}\frac{\partial u_\ell}{\partial x_\ell}\delta_{ij} \equiv 2\mu(T)\left(S_{ij} - \tfrac{1}{3}S_{\ell\ell}\delta_{ij}\right) + \mu_{\mathrm{b}} S_{\ell\ell}\delta_{ij} \quad ; \quad q_i = -\lambda(T)\frac{\partial T}{\partial x_i} \tag{48}$$

Bulk viscosity $\mu_{\mathrm{B}}$ (which is not expected to be important in plane channel flow) was set to 0 (Stokes hypothesis), viscosity $\mu$ follows a Sutherland law,[50] and $\lambda$ follows a modified Sutherland law[51]

$$\mu(T) = \mu_0\left[\frac{T}{T_{\mu_0}}\right]^{\frac{3}{2}}\frac{S_\mu + T_{\mu_0}}{S_\mu + T} \quad ; \quad \lambda(T) = \lambda_0\frac{\mu(T)}{\mu_0}[1 + A_\lambda(T - T_{\mu_0})] \quad ; \quad \mu_{\mathrm{b}} = 0 \tag{49}$$

For the present computations which are concerned with airflow, the various coefficients and constans are:[48,50,51] $R_g = 287.04\ \mathrm{m^2\ s^{-1}\ K^{-1}}$, $\gamma = 1.4$, $\mu_0 \equiv \mu(T_{\mu_0}) = 17.11 \times 10^{-6}$ Pa $s$, $T_{\mu_0} = 273.15$ K, $S_\mu = 110.4$ K, $\lambda_0 \equiv \lambda(T_{\mu_0}) = 0.0242\ \mathrm{W\ m^{-1}\ K^{-1}}$, $A_\lambda = 0.00023\ \mathrm{K^{-1}}$ ($A_\lambda$ was obtained[52] by a least-squares fit of the data in Eckert and Drake[51]).

## B. Plane Channel Flow Configuration and Computational Method

In the particular case of plane channel flow, the boundary surface consists of the upper and lower walls ($\mathfrak{W}_l$ at $y = -\frac{1}{2}L_y$ and $\mathfrak{W}_u$ at $y = \frac{1}{2}L_y$, respectively), and the periodic boundaries at $x = \pm\frac{1}{2}L_x$ and $z = \pm\frac{1}{2}L_z$. DNS computations were run using a high-order low-diffusion solver, described in detail in Gerolymos et al..[35] All the computations used in the present work used an UW9 scheme[53] (reconstruction of the primitive variables[35] without limiters), associated with the HLLC approximate Riemann solver.[54]

## C. Onboard Averages Computation

The averages are computed, as usual, as time-averages of space-averages. Space-averages, for any flow quantity $\mathrm{a}$, are computed over each $xz$-plane at each time-step

$$\overline{\mathrm{a}}^{xz} := \frac{1}{L_x L_z}\int_{-\frac{1}{2}L_x}^{+\frac{1}{2}L_x}\int_{-\frac{1}{2}L_z}^{+\frac{1}{2}L_z}[\mathrm{a}dxdz] \tag{50}$$

**Table 1. Summary of DNS computations.**

| $Re_{\tau_w}$ | $\bar{M}_{\mathrm{B}_w}$ | $\bar{M}_{\mathrm{CL}}$ | $L_x \times L_y \times L_z$ | $N_x \times N_y \times N_z$ | $\overline{\Delta x^+}$ | $\overline{\Delta y_w^+}$ | $\overline{\Delta z^+}$ | scheme |
|---|---|---|---|---|---|---|---|---|
| 182 | 0.300 | 0.341 | $4\pi\delta \times 2\delta \times \frac{4}{3}\pi\delta$ | $57 \times 121 \times 49$ | 52 | 0.2 | 20 | UW9 |
| 185 | 0.300 | 0.347 | $4\pi\delta \times 2\delta \times \frac{4}{3}\pi\delta$ | $57 \times 161 \times 49$ | 52 | 0.2 | 20 | UW9 |
| 185 | 0.300 | 0.347 | $4\pi\delta \times 2\delta \times \frac{4}{3}\pi\delta$ | $121 \times 121 \times 81$ | 24 | 0.2 | 12 | UW9 |
| 239 | 1.507 | 1.492 | $4\pi\delta \times 2\delta \times \frac{4}{3}\pi\delta$ | $57 \times 161 \times 49$ | 52 | 0.2 | 20 | UW9 |
| 239 | 1.507 | 1.502 | $4\pi\delta \times 2\delta \times \frac{4}{3}\pi\delta$ | $121 \times 121 \times 81$ | 24 | 0.2 | 12 | UW9 |

$L_x$, $L_y$, $L_z$ ($N_x$, $N_y$, $N_z$) are the dimensions (number of grid-points) of the computational domain ($x$ = streamwise, $y$ = normal-to-the-wall, $z$ = homogeneous spanwise direction); $\delta$ is the channel halfheight; $\Delta x^+$, $\Delta y_w^+$, $\Delta z^+$ are the mesh-sizes in wall-units; $Re_{\mathrm{B}_w} = \bar{u}_{\mathrm{B}}\delta\bar{\nu}_w^{-1}$; $M_{\mathrm{B}_w} = \bar{u}_{\mathrm{B}}\bar{a}_w^{-1}$; $\bar{u}_{\mathrm{B}}$ = bulk veloctiy; $\bar{\nu}_w$ = kinematic viscosity at the wall; $\bar{a}_w$ = velocity-of-sound at the wall

are computed over each $xz$-plane at each time-step. Time-averages are obtained over an observation-time $t_{\mathrm{OBS}}$

$$\overline{\mathrm{a}}^t := \frac{1}{t_{\mathrm{OBS}}} \int_{t0}^{t0+t_{\mathrm{OBS}}} [\mathrm{a}dt] \tag{51}$$

The reduced size of the computational domain is such that space-averaging over the domain does not give the Reynolds-average (time-average) of the quantity of interest, as would be the case if the computational domain were large enough in the $x$-wise and $z$-wise directions for the ergodic hypothesis to be applicable. These space-averages (Eq. 50) are then time-averaged over a sufficient observation-time $t_{\mathrm{OBS}}$ (Eq. 51) to give the final Reynolds-averages. The corresponding splitting reads

$$\mathrm{a}(x,y,z,t) = \bar{\mathrm{a}}(y) + \mathrm{a}'(x,y,z,t) = \overline{\mathrm{a}}^{xz}(y,t) + \mathrm{a}'^{xz}(x,y,z,t) = \bar{\mathrm{a}}(y) + \underbrace{\mathrm{a}'^t(y,t) + \mathrm{a}'^{xz}(x,y,z,t)}_{\mathrm{a}'(x,y,z,t)} \tag{52}$$

$$\overline{\mathrm{a}}^{xz}(y,t) := \overline{\mathrm{a}(x,y,z,t)}^{xz} \quad ; \quad \overline{\mathrm{a}'^{xz}(x,y,z,t)}^{xz} \equiv 0 \quad ; \quad \overline{\mathrm{a}'(x,y,z,t)}^{xz}(y,t) \equiv \mathrm{a}'^t(y,t) \tag{53}$$

Time-averages are taken over an observation-time $t_{\mathrm{OBS}}$, sufficiently long compared to the characteristic turbulence-timescale $\tau_{\mathrm{T}}$ to have

$$t \gg \tau_{\mathrm{T}} \ : \quad \bar{\mathrm{a}}(y) = \overline{\overline{\mathrm{a}}^{xz}(y,t)}^t \quad ; \quad \overline{\mathrm{a}'^t(y,t)}^t = 0 \tag{54}$$

while even larger observation-times are required (as when computing low-frequency spectra) to achieve

$$t \ggg \tau_{\mathrm{T}} \ : \quad \overline{\mathrm{a}'(x,y,z,t)}^t = \overline{\mathrm{a}'^{xz}(x,y,z,t)}^t = 0 \tag{55}$$

With these splitting (Eq. 52), for any 2 flow quantities $\mathrm{a}$ and $\mathrm{b}$

$$t \gg \tau_{\mathrm{T}} \ : \quad \overline{\mathrm{ab}} = \overline{\overline{\mathrm{ab}}^{xz}}^t = \bar{\mathrm{a}}(y)\bar{\mathrm{b}}(y) + \overline{\mathrm{a}'^t(y,t)\mathrm{b}'^t(y,t)}^t + \overline{\overline{\mathrm{a}'^{xz}\mathrm{b}'^{xz}}^{xz}}^t \tag{56}$$

An onboard (as the computations advance in time) procedure for the computation of the averages is used, where, at every time-step $n_{it} \equiv n$, the correlation $\overline{\mathrm{a}\cdots\mathrm{b}}$ is updated

$$^{(n+1)}[\overline{\mathrm{a}\cdots\mathrm{b}}] = \frac{^{(n+1)}t_{\mathrm{OBS}} - \Delta t}{^{(n+1)}t_{\mathrm{OBS}}} \, ^{(n)}[\overline{\mathrm{a}\cdots\mathrm{b}}] + \frac{\Delta t}{^{(n+1)}t_{\mathrm{OBS}}} \, \overline{\mathrm{a}\cdots\mathrm{b}}^{xz} \tag{57}$$

where $\Delta t$ is the computational time-step, and $^{(n+1)}t_{\mathrm{OBS}} := {}^{(n+1)}t - t_{0_{\mathrm{OBS}}}$ is the observation time at $t = ^{(n+1)} t$.

## D. Green's Function Solution of the Poisson Equations

### 1. *ODEs for the Fourier Transforms*

The originality of the paper is that it post-processing of the DNS-data does not only into separate contributions from the various source-terms in the Poisson equations (Eqs. 43–41), but that these terms are further split into a volume-integral (quasi-homogeneous) part and a surface-integral (strongly inhomogeneous wall-echo)

part. This is achieved, following Kim,[20] by a Green's function approach. In this approach,[20] the generic partial differential equation (PDE)

$$\nabla^2 p'_{(m)} = Q'_{(m)} \quad ; \quad y = \pm\tfrac{1}{2}L_y \ : \ \frac{\partial p'_{(m)}}{\partial y} = B'_{(m)_\pm}(x,z,t) \tag{58}$$

where $(m) = [(\tau),(\mathrm{BF}),(s),(\rho' s),(r),(\rho' r),(\rho'),(\Theta),(\dot{V}\nabla\bar{\rho}),(\dot{V}\nabla\rho')]$, is reduced, taking into account the fact that the directions $x$ and $z$ are homogeneous, to an ordinary differential equation (ODE) for the $xz$-Fourier-transform of $p'_{(m)}(x,y,z,t)$

$$\left[\frac{d^2}{dy^2} - \kappa^2\right] \hat{p}'_{(m)}(\kappa_x,y,\kappa_z,t) = \hat{Q}'_{(m)}(\kappa_x,y,\kappa_z,t) \quad ; \quad y = \pm\tfrac{1}{2}L_y \ : \ \frac{\partial \hat{p}'_{(m)}}{\partial y} = \hat{B}'_{(m)_\pm}(\kappa_x,\kappa_z,t) \tag{59}$$

$$\kappa^2 := \kappa_x^2 + \kappa_z^2 \tag{60}$$

with

$$p'_{(m)}(x,y,z,t) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \hat{p}'_{(m)}(\kappa_x,y,\kappa_z,t)\ e^{i\kappa_x x + i\kappa_z z}\ d\kappa_x\ d\kappa_z \tag{61}$$

$$Q'_{(m)}(x,y,z,t) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \hat{Q}'_{(m)}(\kappa_x,y,\kappa_z,t)\ e^{i\kappa_x x + i\kappa_z z}\ d\kappa_x\ d\kappa_z \tag{62}$$

$$B'_{(m)_\pm}(x,z,t) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \hat{B}'_{(m)_\pm}(\kappa_x,\kappa_z,t)\ e^{i\kappa_x x + i\kappa_z z}\ d\kappa_x\ d\kappa_z \tag{63}$$

The Fourier-trasforms $\hat{p}'_{(m)}$, $\hat{Q}'_{(m)}$, and $\hat{B}'_{(m)_\pm}$ are computed using standard DFT (discrete Fourier transform) techniques,[55] in the periodic directions $x$ and $z$, with maximum computable wavenumbers $\kappa_{x_{\max}} = \pi(\Delta x)^{-1}$ and $\kappa_{z_{\max}} = \pi(\Delta z)^{-1}$. Notice, that for the plane channel flow considered here, $f_{\mathrm{V}_y} \equiv 0$, so that only $B_{(\tau)_\pm} \neq 0$, all the other terms having homogeneous Neumann boundary-conditions[27] ($B_{(m)_\pm} = 0\ \forall (m) \neq (\tau)$).

### 2. *Green's Functions*

$\underline{\kappa \neq 0}$: The ODEs for the various contributions to the pressure fluctuations are computed using standard Green's function techniques[23–28]

$$\hat{p}'_{(m)}(\kappa_x,y,\kappa_z,t) = \left\{ \int_{-\frac{1}{2}L_y}^{+\frac{1}{2}L_y} \left[G_{\mathrm{Kim}}(\kappa,y,Y)\ \hat{Q}'_{(m)}(\kappa_x,y,\kappa_z,t)\right] dY \right\} + \hat{p}'_{(m)_{\mathrm{BCs}}}(\kappa_x,y,\kappa_z,t) \tag{64}$$

$$\hat{p}'_{(m;\mathfrak{V})}(\kappa_x,y,\kappa_z,t) = \left\{ \int_{-\frac{1}{2}L_y}^{+\frac{1}{2}L_y} \left[G_\infty(\kappa,y,Y)\ \hat{Q}'_{(m)}(\kappa_x,y,\kappa_z,t)\right] dY \right\} \tag{65}$$

$$\hat{p}'_{(m;w)}(\kappa_x,y,\kappa_z,t) = \hat{p}'_{(m)}(\kappa_x,y,\kappa_z,t) - \hat{p}'_{(m;\mathfrak{V})} \tag{66}$$

where

$$\left[\frac{d^2}{dy^2} - \kappa^2\right] G_{\mathrm{Kim}}(\kappa,y,Y) = \delta(y-Y) \quad ; \quad \left[\frac{dG_{\mathrm{Kim}}}{dy}\right](\kappa, y = \pm\tfrac{1}{2}L_y, Y) = 0 \tag{67}$$

$$\left[\frac{d^2}{dy^2} - \kappa^2\right] \hat{p}'_{(m)_{\mathrm{BCs}}}(\kappa_x,y,\kappa_z,t) = 0 \quad ; \quad \left[\frac{d}{dy}\hat{p}'_{(m)_{\mathrm{BCs}}}\right](\kappa_x, y = \pm\tfrac{1}{2}L_y, \kappa_z, t) = \hat{B}'_{(m)_\pm}(\kappa_x,\kappa_z,t) \tag{68}$$

$$\left[\frac{d^2}{dy^2} - \kappa^2\right] G_\infty(\kappa,y,Y) = \delta(y-Y) \quad ; \quad \lim_{|y-Y|\longrightarrow\infty} G_\infty(\kappa,y,Y) = 0 \tag{69}$$

The relation for $\hat{p}'_{(m;w)}$ (Eq. 66) is obtained by analogy to (Eqs. 2, 3), and taking into account the fact that surface integrals at the $x = \pm\frac{1}{2}L_x$ and $z = \pm\frac{1}{2}L_z$ boundaries cancel out because of periodicity. Kim[20] solved the problem for $G_{\mathrm{Kim}}$ (Eq. 67). The nonhomogeneous gradient-boundary-conditions term for $p_{(\tau)'}$ can be taken into account by simply superposing an appropriate boundary-conditions term (obtained[10,29] by straightforward integration of Eq. 68), to the Green's function solution (Eq. 64). Finally, the terms corresponding to

volume-integrals[18] (by analogy with [Eqs. 2, 3]) are computed using the free-space Green's function $G_\infty$ (Eq. 69). When $\kappa \neq 0$ the solutions are obtained by straightforward application of standard methods[25,28]

$$G_{\rm Kim}(\kappa \neq 0, y, Y) = -\frac{\cosh[\kappa(L_y - |y-Y|)] + \cosh[\kappa(y+Y)]}{2\kappa \sinh \kappa L_y} \tag{70}$$

$$\kappa := \sqrt{\kappa_x^2 + \kappa_z^2} \neq 0 \ : \ \hat{p}'_{(m)_{\rm BCs}}(\kappa_x, y, \kappa_z, t) = \frac{B'_{(m)_+} \cosh[\kappa(\frac{1}{2}L_y + y)] - B'_{(m)_-} \cosh[\kappa(\frac{1}{2}L_y - y)]}{\kappa \sinh \kappa L_y} \tag{71}$$

$$G_\infty(\kappa \neq 0, y, Y) = -\frac{e^{-\kappa|y-Y|}}{2\kappa} \tag{72}$$

Notice that the present expression for $G_{\rm Kim}(\kappa \neq 0, y, Y)$ (Eq. 70) can be shown, using hyperbolic trigonometric identities, to be equivalent to the expression given in (Eq. 7) of Kim.[20] It was preferred, not only because it offers an if-less construction, but also because it clearly brings forward the relation $G(y,Y) = G(Y,y)$, as expected because of the selfadjointedness of the the operator $[d_y^2(\cdot) - \kappa^2]$.[24] Notice also that $G_\infty(\kappa \neq 0, y, Y) = \lim_{L_y \longrightarrow \infty} G_{\rm Kim}(\kappa \neq 0, y, Y)$.

<u>$\kappa = 0$:</u> The case $\kappa = 0 \Longrightarrow \kappa_x = \kappa_z = 0$ is more complex, and merits being discussed in some detail. Obviously, the limits as $\kappa \longrightarrow 0$ of the $\kappa \neq 0$ solutions (Eqs. 70–72) are singular, contrary to the case of the same problem with Dirichlet boundary-conditions.[27] This is due to the fact that solutions for the corresponding $\kappa = \kappa_x = \kappa_z = 0$ equations for the Green's functions (obtained from [Eqs. 67–69]) with the associated Neumann boundary-conditions[27] (Eqs. 70–69) cannot be found. Indeed, the existence of solution for the original equation (Eq. 59) with $\kappa = 0$ is conditional. The equation

$$\frac{d^2\,\hat{p}'_{(m)}}{dy^2}(\kappa_x = 0, y, \kappa_z = 0, t) = \hat{Q}_{(m)}(\kappa_x = 0, y, \kappa_z = 0, t) \quad ; \quad y = \pm\tfrac{1}{2}L_y \ : \ \frac{\partial \hat{p}'_{(m)}}{\partial y} = 0 \tag{73}$$

can be integrated once to give

$$\int_{-\frac{1}{2}L_y}^{+\frac{1}{2}L_y} \hat{Q}_{(m)}(\kappa_x = 0, Y, \kappa_z = 0, t)\, dY = 0 \tag{74}$$

indicating that (Eq. 73) with homogeneous Dirichlet boundary-conditions[27] $\hat{B}'_{(m)_\pm} = 0$ (recall that $\forall (m) \neq (\tau) \ : \ \hat{B}_{(m)_\pm} = 0$) admits any constant function as solution (any constant function is an eigenfunction[23,24,28] of [Eq. 73], and $\kappa = 0$ is a characteristic number of [Eq. 59] with boundary-conditions $\hat{B}'_{(m)_\pm} = 0$). Therefore a solution only exists provided the corresponding compatibility relation[23,24] holds, *ie* $\hat{Q}$ is orthogonal to the constant function (Eq. 74). This analogous to the 3-D compatibility relation for the Poisson equation with Neumann boundary-conditions.[26,27]

Provided the compatibility relations (Eqs. 74) hold, the solution for $\kappa = 0$ with boundary-conditions $\hat{B}_{(m)_\pm} = 0$ can be obtained using the Green's function solution (Eqs. 64–66) with the $\kappa = 0$ free-space Green's function

$$G_{\rm Kim}(\kappa = 0, y, Y) = G_\infty(\kappa = 0, y, Y) = \frac{|y-Y|}{2} \tag{75}$$

and

$$\hat{p}'_{(m)_{\rm BCs}}(\kappa_x = 0, y, \kappa_z = 0, t) = 0 \tag{76}$$

As a consequence,

$$\hat{p}'_{(m;\mathfrak{V})}(\kappa_x = 0, y, \kappa_z = 0, t) = \hat{p}'_{(m)}(\kappa_x = 0, y, \kappa_z = 0, t) \quad \Longleftrightarrow \quad \hat{p}'_{(m;w)}(\kappa_x = 0, y, \kappa_z = 0, t) = 0 \tag{77}$$

The $\kappa = 0$ free-space Green's function is obtained by directly solving $d_y^2 G_\infty(\kappa = 0, y, Y) = \delta(y - Y)$ and requiring[18] $G_\infty(\kappa = 0, y, Y) = G_\infty(\kappa = 0, |y-Y|) \propto |y-Y|$. Notice that in this case it is the compatibility relation (Eq. 74), and not the Green's function which is responsible for satisfying the boundary-conditions since

$$\begin{aligned}\left[\frac{d\,\hat{p}'_{(m)}}{dy}\right](\kappa_x = 0, y, \kappa_z = 0, t) &= \frac{d}{dy}\left\{\int_{-\frac{1}{2}L_y}^{+\frac{1}{2}L_y} \left[\frac{|y-Y|}{2}\,\hat{Q}'_{(m)}(\kappa_x = 0, y, \kappa_z = 0, t)\right] dY\right\} \\ &= \int_{-\frac{1}{2}L_y}^{+\frac{1}{2}L_y} \left[\frac{\mathrm{sign}(y-Y)}{2}\,\hat{Q}'_{(m)}(\kappa_x = 0, Y, \kappa_z = 0, t)\right] dY = 0\end{aligned} \tag{78}$$

with $\mathrm{sign}(\pm\frac{1}{2}L_y - Y) = \pm 1 \;\forall\; Y \in [-\frac{1}{2}L_y, +\frac{1}{2}L_y]$. Finally, the solution of the boundary-conditions problem (required for $p'_{(\tau)}$)

$$\frac{d^2\,\hat{p}'_{(m)_{\mathrm{BCs}}}}{dy^2}(\kappa_x = 0, y, \kappa_z = 0, t) = 0 \quad ; \quad y = \pm\tfrac{1}{2}L_y \;:\; \frac{\partial \hat{p}'_{(m)_{\mathrm{BCs}}}}{\partial y} = \hat{B}'_{(m)_\pm}(\kappa_x = 0, \kappa_z = 0, t) \tag{79}$$

which is obviously a linear function of $y$ exists iff (if and only if)

$$\left\{\exists\, \hat{p}'_{(m)_{\mathrm{BCs}}}(\kappa_x = 0, y, \kappa_z = 0, t) \;:\; \{\text{Eq. 77}\}\right\} \quad \Longleftrightarrow \quad \hat{B}'_{(m)_-}(\kappa_x = 0, \kappa_z = 0, t) = \hat{B}'_{(m)_+}(\kappa_x = 0, \kappa_z = 0, t) \tag{80}$$

### 3. *Adaptation to Channel Flow DNS*

The Green's function solutions and the boundary-conditions corrections for the wavenumbers $\kappa \neq 0$ (Eqs. 64–72) are directly applicable. To compute the contribution of the wavenumber $\kappa = 0$, one should bear in mind that $\kappa = 0$ corresponds to space-averaging (Eq. 50), so that

$$\hat{Q}'_{(m)}(\kappa_x = 0, y, \kappa_z = 0, t) \equiv \overline{Q'_{(m)}(x, y, z, t)}^{xz} \equiv Q'^{t}_{(m)}(y, t) \tag{81}$$

$$\begin{aligned}\hat{B}'_{(m)_\pm}(\kappa_x = 0, \kappa_z = 0, t) &\equiv \overline{B'_{(m)_\pm}(x, z, t)}^{xz} \equiv \overline{\left[\frac{\partial p'_{(m)}}{\partial y}\right](x, y = \pm\tfrac{1}{2}L_y, z, t)}^{xz} \\ &\equiv \left[\frac{\partial}{\partial y}\overline{p'_{(m)}}^{xz}\right](y = \pm\tfrac{1}{2}L_y, t) \equiv \frac{\partial p'^{t}_{(m)}}{\partial y}(y = \pm\tfrac{1}{2}L_y, t)\end{aligned} \tag{82}$$

Therefore

$$\frac{1}{L_y}\int_{-\frac{1}{2}L_y}^{+\frac{1}{2}L_y} \hat{Q}'_{(m)}(\kappa_x = 0, y, \kappa_z = 0, t)\; dy \equiv \frac{1}{L_x L_y L_z}\int_{-\frac{1}{2}L_x}^{+\frac{1}{2}L_x}\int_{-\frac{1}{2}L_y}^{+\frac{1}{2}L_y}\int_{-\frac{1}{2}L_z}^{+\frac{1}{2}L_z} Q'_{(m)}(x, y, z, t)\; dx\; dy\; dz \equiv Q'_{(m;\mathrm{B})}(t) \tag{83}$$

is the bulk fluctuation of $Q_{(m)}$ at a given time. In general bulk fluctuations are small, and would be identically 0 if the computational domain were large enough for the ergodic hypothesis to apply on space-averages. We therefore slightly modify the $Q'_{(m)}$ and $B'_{(\tau)_\pm}$ terms obtained from the DNS computations processing

$$Q'_{(m)} \models [Q'_{(m)}]_{\mathrm{DNS}} - Q'_{(m;\mathrm{B})} \;\Longleftrightarrow$$

$$\hat{Q}'_{(m)}(\kappa_x = 0, y, \kappa_z = 0, t) \models [\hat{Q}'_{(m)}]_{\mathrm{DNS}}(\kappa_x = 0, y, \kappa_z = 0, t) - Q'_{(m;\mathrm{B})}(t) \tag{84}$$

$$\hat{B}'_{(\tau)}(\kappa_x = 0, \kappa_z = 0, t) \models 0 \tag{85}$$

Notice that this modification of $Q'_{(m)}$, which ensures that the compatibility relation (Eq. 74) holds, so that the solution of (Eq. 73) exists, allows for slightly time-fluctuating space-averages at different $y$-positions, because of the limited computational domains used in the simulations. The associated small numerical error can be further reduced by increasing the size of the computational domain (increasing $L_x$ and $L_z$, for a given $L_y$).

### 4. *Onboard Pressure-Splitting and Correlations*

An onboard postprocessing module is developed to apply the pressure-splitting methodology. The Green's function solution methodology (§III.D) is applied at every time-step of the computations, yielding instantaneous 3-D fields of the various terms retained in the pressure-splitting (Eqs. 43–41), each of which is further split into volume and wall-echo parts

$$p' = \sum_{m=1} M p'_{(m)} = \sum_{m=1} M\left[p'_{(m;\mathfrak{V})} + p'_{(m;w)}\right] \tag{86}$$

$$\tag{87}$$

All the correlations which are linear in fluctuting pressure, such as $\phi_{ij}$, $\overline{p'u'_i}$, $\Pi_{ij}$, can then be linearly split in the same way.

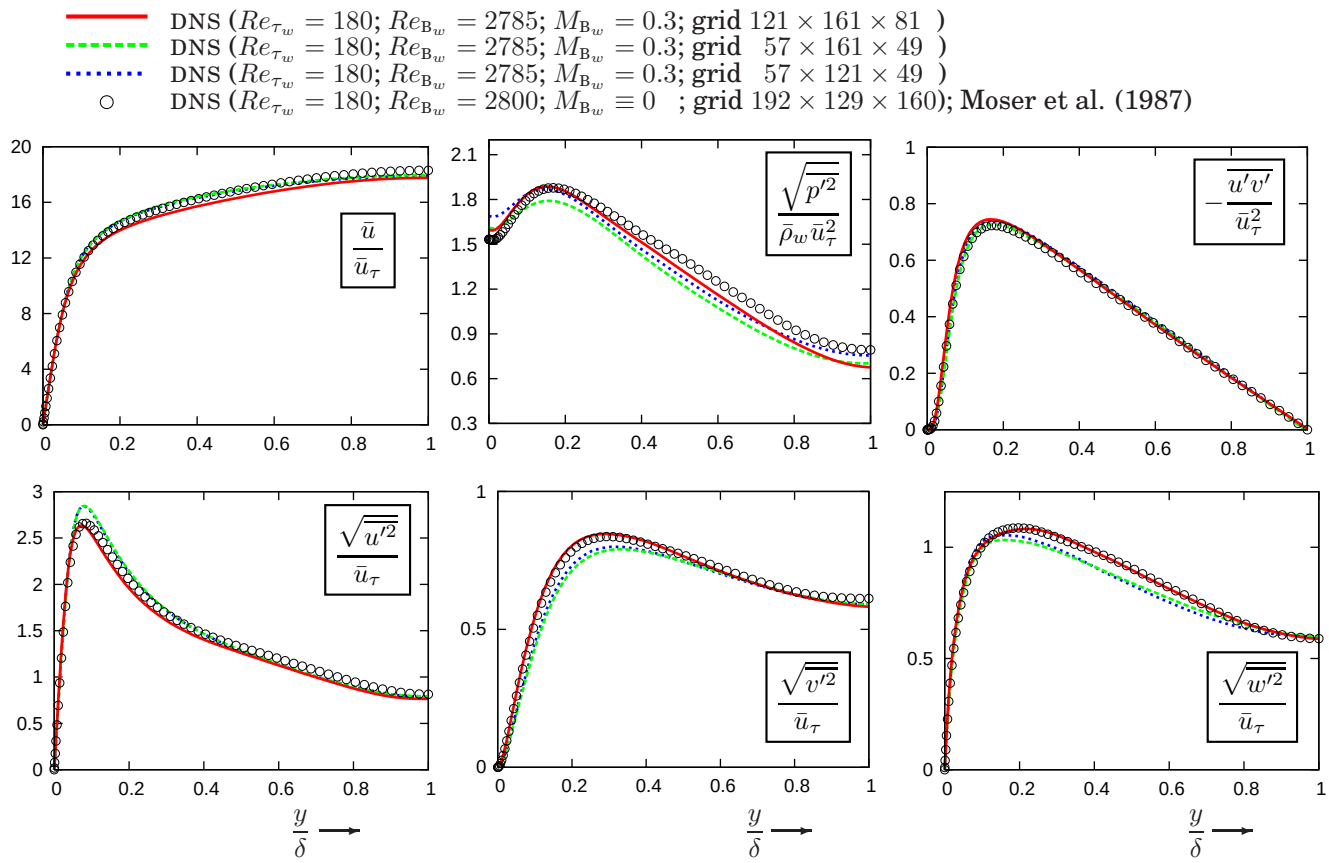


**Figure 1. Comparison of present DNS-computed statistics ($Re_{\tau_w}$ = 185; $M_{B_w}$ = 0.3; $\bar{M}_{CL}$ = 0.35) with incompressible DNS results of Kim et al.[21,22] ($Re_{\tau_w}$ = 180; $M_B \equiv 0$).**

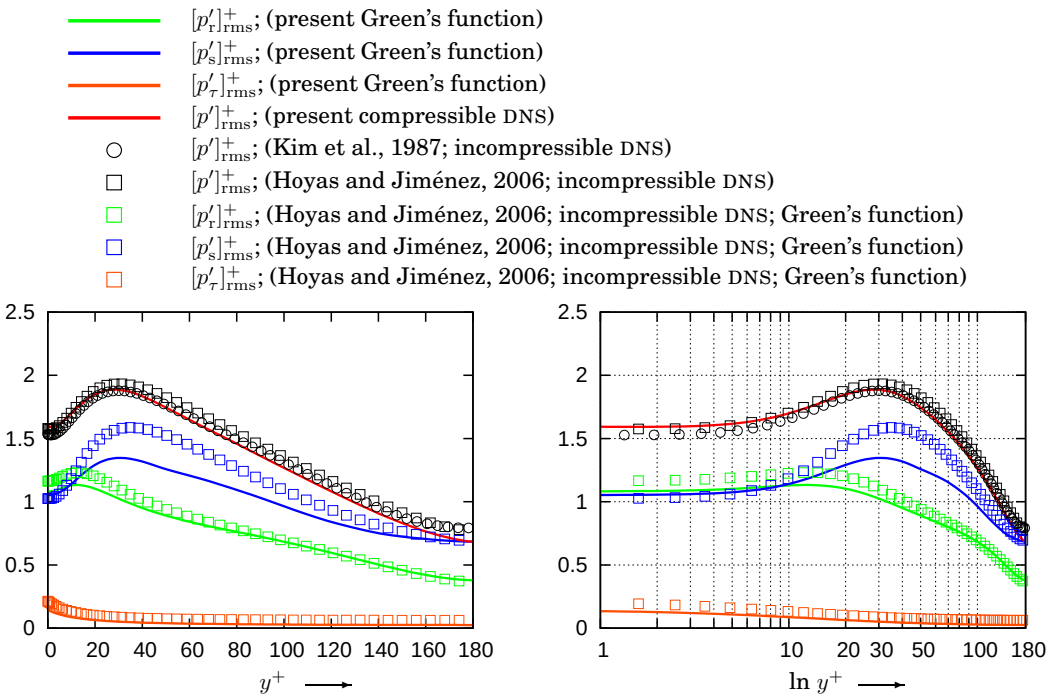


**Figure 2. Comparison of rms-values of $p'$, $p'_{(r)}$, $p'_{(s)}$, and $p'_{(\tau)}$, from preliminary results of the present DNS computations ($Re_{\tau_w}$ = 185; $M_{B_w}$ = 0.3; $\bar{M}_{CL}$ = 0.35; grid 121 × 161 × 81; for a very short observation-time $t^+_{OBS}$ = 82) with incompressible DNS results of Hoyas and Jimenez[31] ($Re_{\tau_w}$ = 180; $M_B \equiv 0$).**

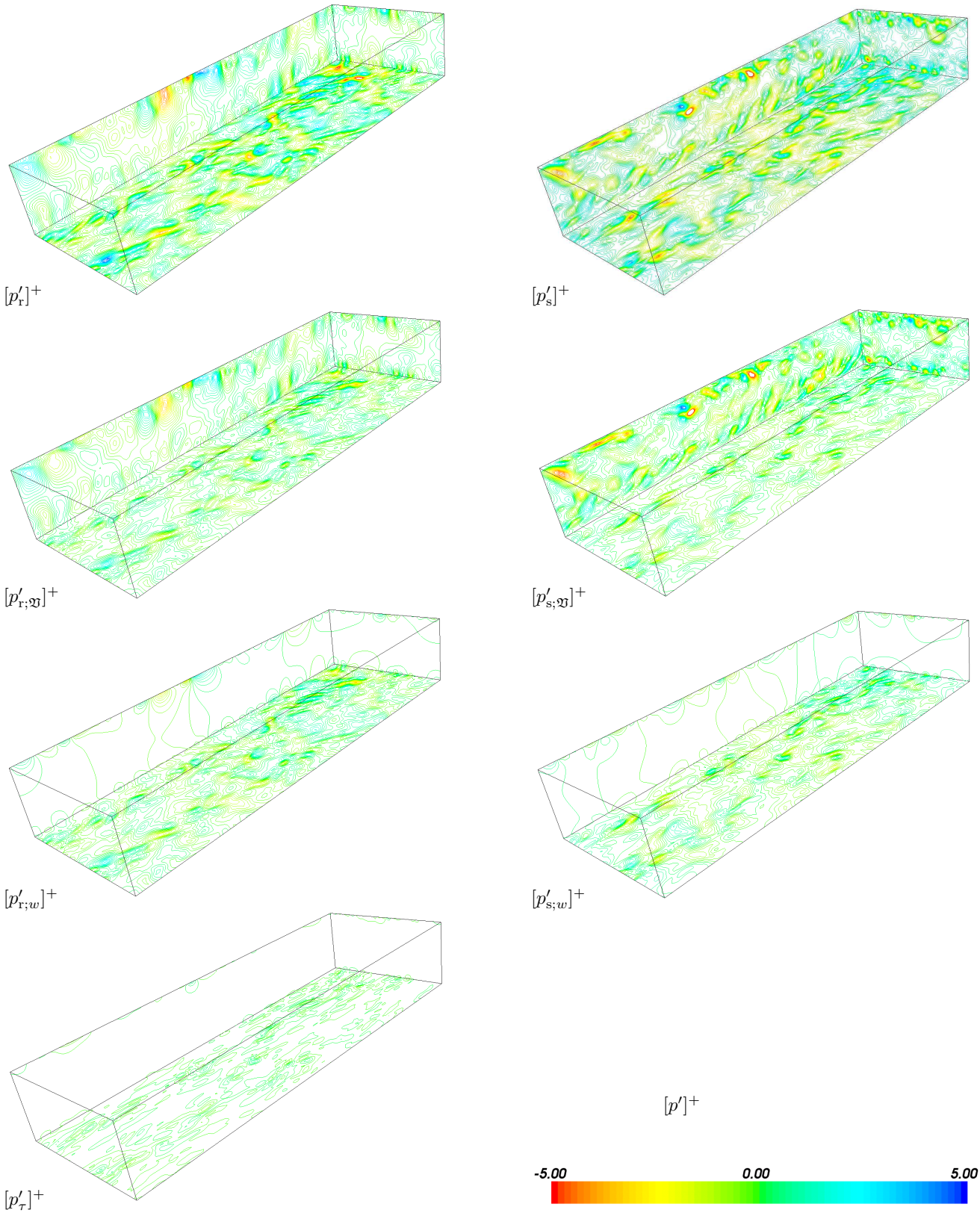


**Figure 3. Instantaneous fluctuating-pressure fields using the present $p'$-decomposition, from DNS-computations ($Re_{\tau_w} = 185$; $M_{\mathrm{B}_w} = 0.3$; $\bar{M}_{\mathrm{CL}} = 0.35$; grid $121 \times 161 \times 81$), on the lower-wall ($y = -\frac{1}{2}L_y$), at the outflow-boudary ($x = +\frac{1}{2}L_z$) and on the spanwise-periodicity-boundary ($z = -\frac{1}{2}L_z$).**

## V. Quasi-Incompressible Flow Turbulent Correlations

The initial investigation concerned a ($Re_{\tau_w} = 185$; $M_{\mathrm{B}} = 0.3$; $\bar{M}_{\mathrm{CL}} = 0.35$) flow (Fig. 1), for which one may reasonably neglect density fluctuations. In that case the $p'$-splitting was performed using the quasi-incompressible form (Eqs. 43–46). The averages for the $p'$-splitting results presented in the present paper were obtained for a very short observation time $t^+_{\mathrm{OBS}} = 82$. Therefore the data given in the paper should be considered as preliminary results used to illustrate the $p'$-splitting into volume and wall-echo parts.

### A. Fluctuating Pressure Field

The unsplit $p'_{\mathrm{rms}}$ obtained from the compressible DNS computations, agrees quite well with existing incompressible data (Fig. 2). Despite the very short observation time, $[p'_{(\mathrm{r})}]_{\mathrm{rms}}$, $[p'_{(\mathrm{s})}]_{\mathrm{rms}}$, and $[p'_{(\tau)}]_{\mathrm{rms}}$ agree well with the incompressible DNS results of Hoyas and Jimenez[31] (Fig. 3). Nonetheless, the $[p'_{(\mathrm{s})}]_{\mathrm{rms}}$ is substantially lower thna the value obtained by Hoyas and Jimenez.[31] This is related mainly to the insufficient $t^+_{\mathrm{OBS}}$, but could also be partly attributed to the fact that for the present preliminary calculations, the source terms (Eqs. 28, 43–46) were evaluated using a C2 scheme (centered second-order). Future computation will use a C8 scheme (centered 8th-order) for the evaluation of the source terms.

Further splitting of $p'$ into volume and wall-echo contributions (Fig. 4), using the present Green's functions, indicates that the volume and wall-echo contributions, both rapid and slow, are approximately equal at the wall.

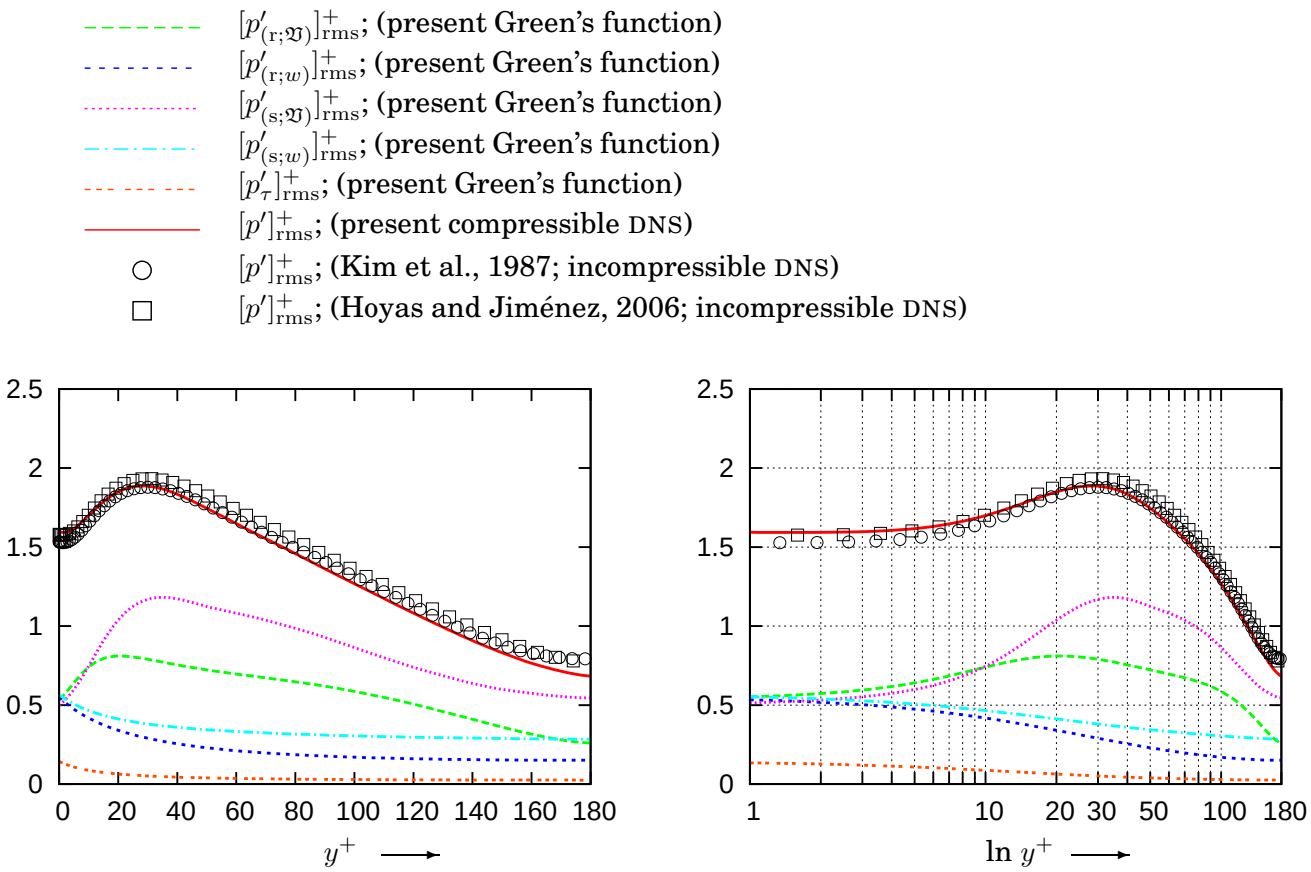


**Figure 4. Distributions of** rms**-values of the 5 terms in the quasi-incompressible** $p'$**-splitting,** ($p'_{(\mathrm{r};\mathfrak{V})}$, $p'_{(\mathrm{r};w)}$, $p'_{(\mathrm{s};\mathfrak{V})}$, $p'_{(\mathrm{s};w)}$, **and** $p'_{(\tau)}$**, from preliminary results of the present** DNS **computations** ($Re_{\tau_w} = 185$; $M_{\mathrm{B}_w} = 0.3$; $\bar{M}_{\mathrm{CL}} = 0.35$; **grid** $121 \times 161 \times 81$; **for a very short observation-time** $t^+_{\mathrm{OBS}} = 82$).

The level of the volume-terms ($p'_{(\mathrm{r};\mathfrak{V})}$, $p'_{(\mathrm{s};\mathfrak{V})}$), at the wall are approximately equal one with another, but also with the corresponding wall-echo terms ($p'_{(\mathrm{r};w)}$, $p'_{(\mathrm{s};\mathfrak{V})}$), highlighting the importance of the echo-terms in turbulence modelling. The wall-echo terms remain important in the entire channel. Nonetheless, as will be seen in the following, they are correlated with the velocity field only in the immediate vicinity of the wall. The Stokes pressure $[p'_{(\tau)}]_{\mathrm{rms}}$ is, expectedly substantially lower than the other terms, being $O[Re^{-1}]$.

## B. Pressure Diffusion

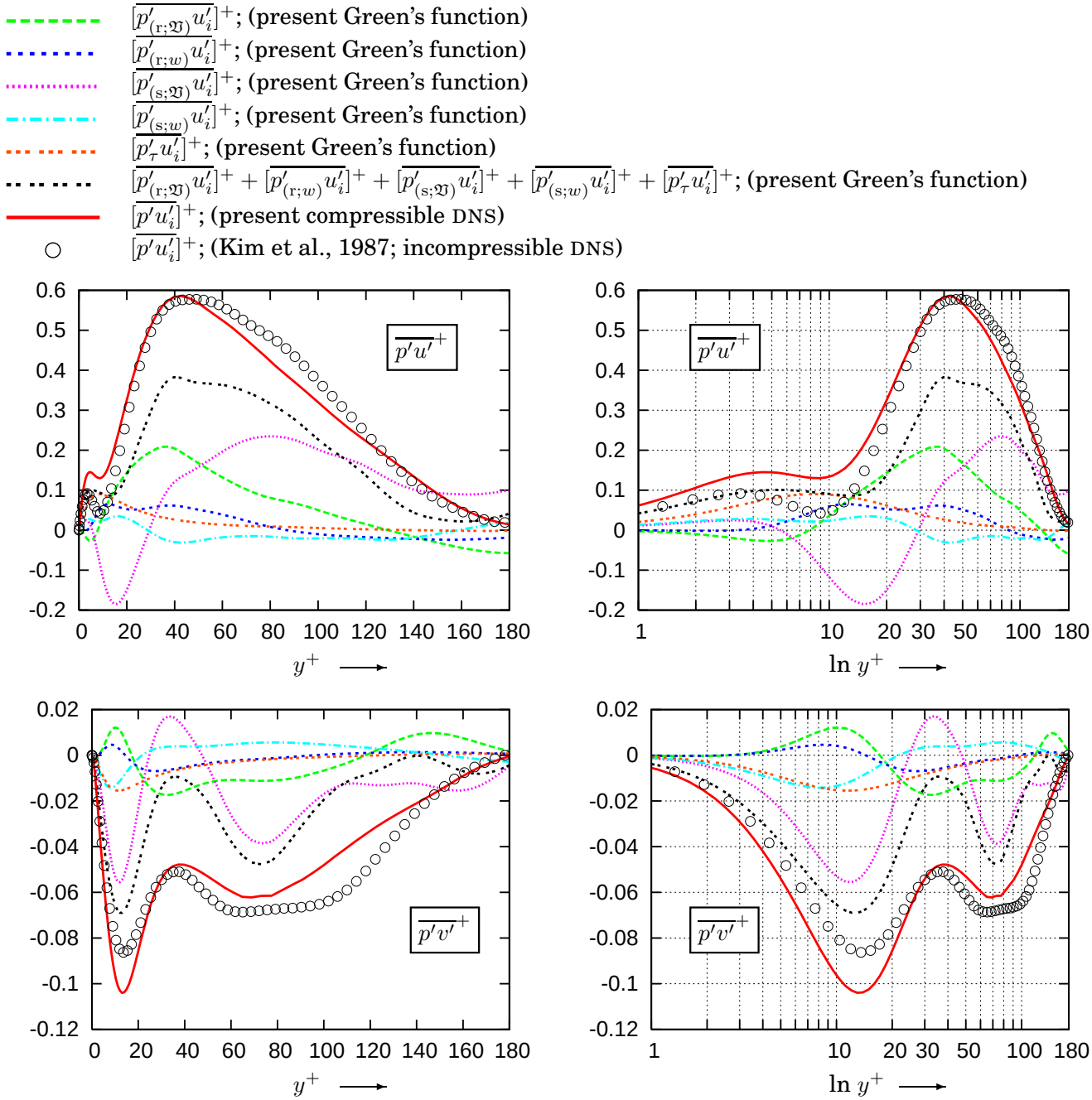


**Figure 5. Distributions of $\overline{p'u'_i}$ for the 5 terms in the quasi-incompressible $p'$-splitting, ($\overline{p'_{(\mathrm{r};\mathfrak{V})}u'_i}$, $\overline{p'_{(\mathrm{r};w)}u'_i}$, $\overline{p'_{(\mathrm{s};\mathfrak{V})}u'_i}$, $\overline{p'_{(\mathrm{s};w)}u'_i}$, and $\overline{p'_{(\tau)}u'_i}$, from preliminary results of the present DNS computations ($Re_{\tau_w} = 185$; $M_{\mathrm{B}_w} = 0.3$; $\bar{M}_{\mathrm{CL}} = 0.35$; grid $121 \times 161 \times 81$; for a very short observation-time $t^+_{\mathrm{OBS}} = 82$).**

The pressure-transport $\overline{p'u'_i}$ vector is split accordingly (Fig. 5). The wall echo-terms are comparable ot the volume terms, at least for $y^+ \leq 10$. This suggests that models for pressure diffusion should include appropriate wall-echo terms, as noted by Vallet.[33] Near the wall, the Stokes term is important compared to the other terms for this low-Reynolds-number flow. The wall-echo terms are quite small for $y^+ \geq 20$ (Fig. 5), indicating the $p'_{(\tau)}$ (Fig. 4) is not well correlated with $u'_i$ away from the wall.

## C. Redistribution $\phi_{ij}$

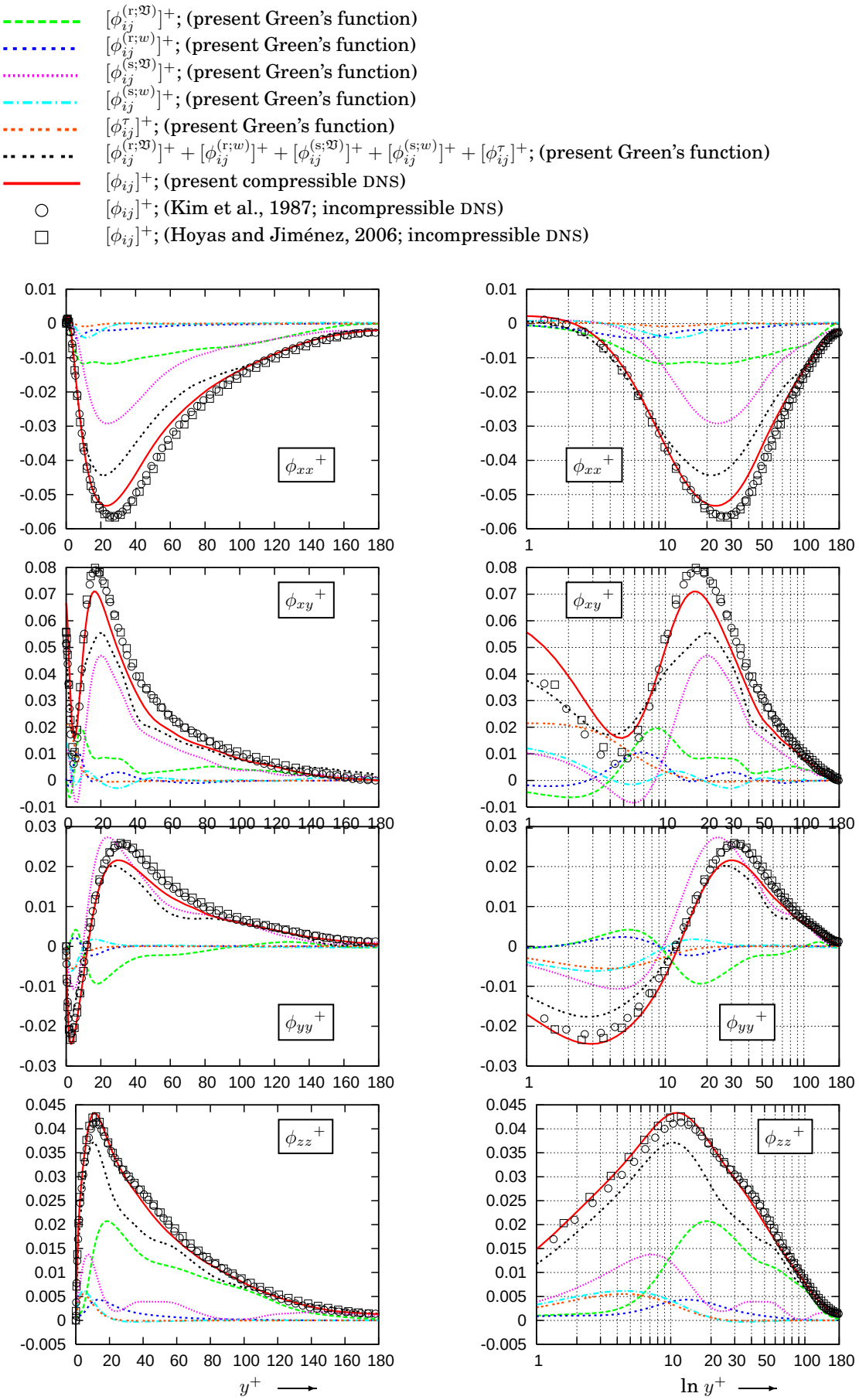


**Figure 6. Distributions of $\phi_{ij}$ for the 5 terms in the quasi-incompressible $p'$-splitting, ($\phi_{ij}^{(\mathrm{r};\mathfrak{V})}$, $\phi_{ij}^{(\mathrm{r};w)}$, $\phi_{ij}^{(\mathrm{s};\mathfrak{V})}$, $\phi_{ij}^{(\mathrm{s};w)}$, and $\phi_{ij}^{(\tau)}$, from preliminary results of the present DNS computations ($Re_{\tau_w} = 185$; $M_{\mathrm{B}_w} = 0.3$; $\bar{M}_{\mathrm{CL}} = 0.35$; grid $121 \times 161 \times 81$; for a very short observation-time $t^+_{\mathrm{OBS}} = 82$).**

The redistribution tensor $\phi_{ij}$ is also split into five terms and the wall-echo terms are important for $y^+ \leq 30$ (Fig. 6). Notice how the Stokes term $\phi_{ij}^{(\tau)}$ is comparable to the other four terms near the wall ($y^+ \leq 10$, Fig. 6). The homogeneous and inhomogeneous terms modelling of $\phi_{ij}$ is essential in the development of RSMs. Obviously, $\phi_{ij}^{(\mathrm{s};w)}$ and $\phi_{ij}^{(\mathrm{r};w)}$ should be related to the inhomogeneous part. Nonetheless, the volume terms contain both a homogeneous and an inhomogeneous part, and the use of the present splitting for modelling purposes should take this into account.

## D. Convergence of the Averages

The previous results were obtained for a quite limited $t^+_{\mathrm{OBS}}$, and should be considered as a preliminary illustration of the above splitting.

- $[\overline{p'_{(\mathrm{r};\mathfrak{V})}u'_i}]^+ + [\overline{p'_{(\mathrm{r};w)}u'_i}]^+ + [\overline{p'_{(\mathrm{s};\mathfrak{V})}u'_i}]^+ + [\overline{p'_{(\mathrm{s};w)}u'_i}]^+ + [\overline{p'_{\tau}u'_i}]^+$; (Green's function); $t^+_{\mathrm{OBS}} = 21$
- $[\overline{p'_{(\mathrm{r};\mathfrak{V})}u'_i}]^+ + [\overline{p'_{(\mathrm{r};w)}u'_i}]^+ + [\overline{p'_{(\mathrm{s};\mathfrak{V})}u'_i}]^+ + [\overline{p'_{(\mathrm{s};w)}u'_i}]^+ + [\overline{p'_{\tau}u'_i}]^+$; (Green's function); $t^+_{\mathrm{OBS}} = 52$
- $[\overline{p'_{(\mathrm{r};\mathfrak{V})}u'_i}]^+ + [\overline{p'_{(\mathrm{r};w)}u'_i}]^+ + [\overline{p'_{(\mathrm{s};\mathfrak{V})}u'_i}]^+ + [\overline{p'_{(\mathrm{s};w)}u'_i}]^+ + [\overline{p'_{\tau}u'_i}]^+$; (Green's function); $t^+_{\mathrm{OBS}} = 82$
- $[\overline{p'u'_i}]^+$; (present compressible DNS)
- $[\overline{p'u'_i}]^+$; (Kim et al., 1987; incompressible DNS)

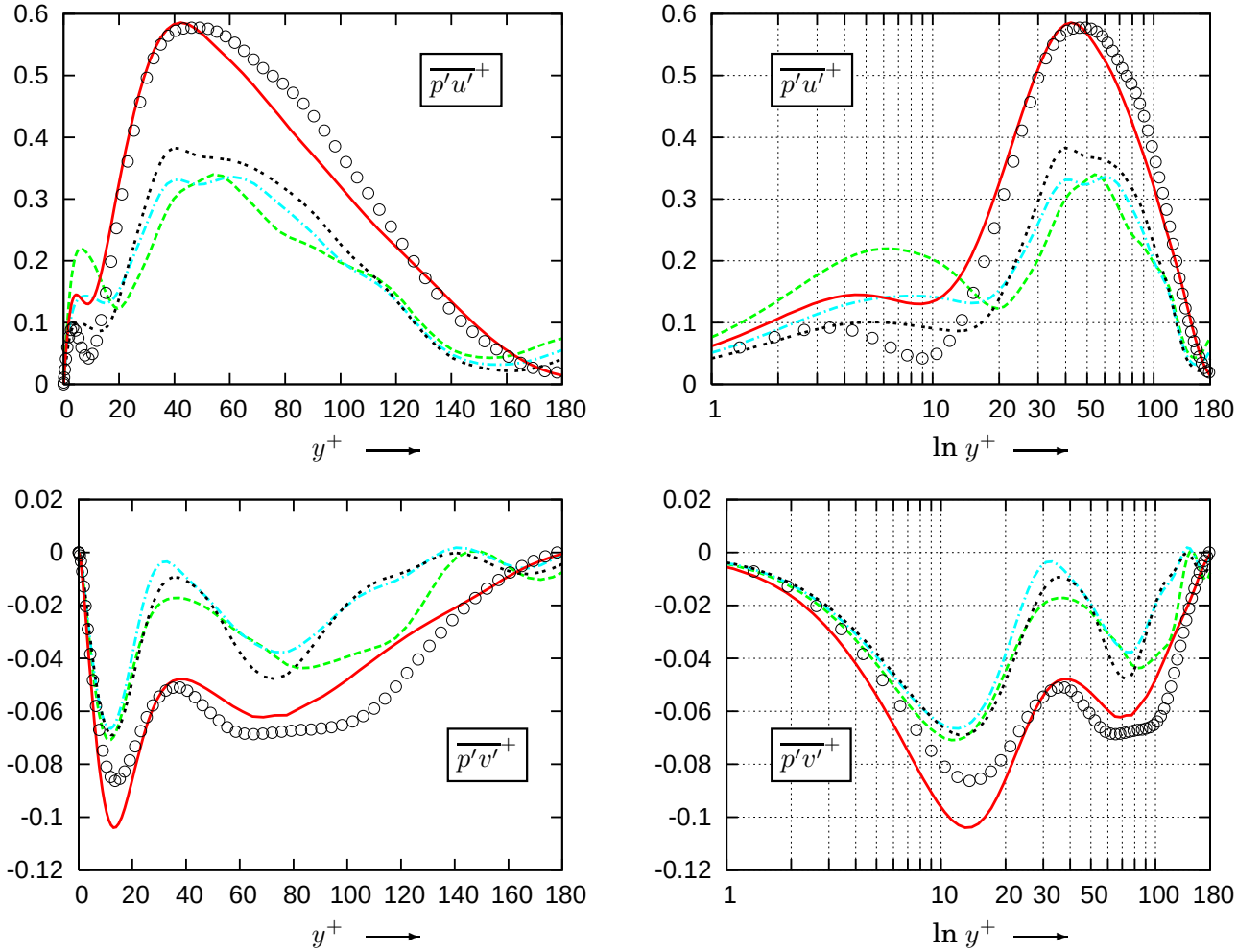


**Figure 7. Convergence of the distributions of the sum of the 5 terms in the quasi-incompressible $p'$-splitting, ($\overline{p'_{(\mathrm{r};\mathfrak{V})}u'_i}$, $\overline{p'_{(\mathrm{r};w)}u'_i}$, $\overline{p'_{(\mathrm{s};\mathfrak{V})}u'_i}$, $\overline{p'_{(\mathrm{s};w)}u'_i}$, and $\overline{p'_{(\tau)}u'_i}$, from preliminary results of the present DNS computations ($Re_{\tau_w} = 185$; $M_{\mathrm{B}_w} = 0.3$; $\bar{M}_{\mathrm{CL}} = 0.35$; grid $121 \times 161 \times 81$; up to a very short observation-time $t^+_{\mathrm{OBS}} = 82$).**

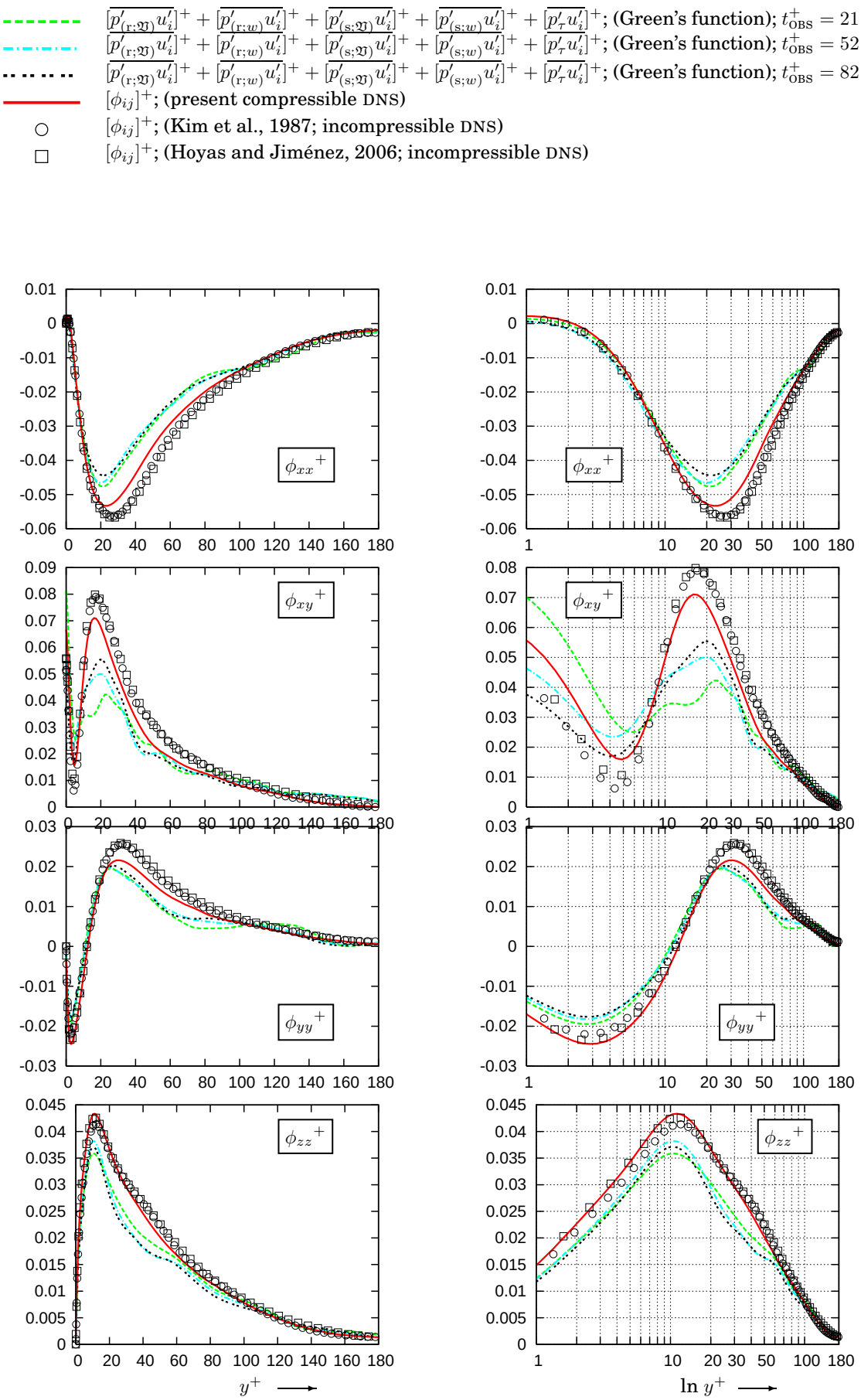


**Figure 8. Convergence of the distributions of the sum of the 5 terms in the quasi-incompressible $p'$-splitting, ($\phi_{ij}^{(\mathrm{r};\mathfrak{V})}$, $\phi_{ij}^{(\mathrm{r};w)}$, $\phi_{ij}^{(\mathrm{s};\mathfrak{V})}$, $\phi_{ij}^{(\mathrm{s};w)}$, and $\phi_{ij}^{(\tau)}$, from preliminary results of the present DNS computations ($Re_{\tau_w} = 185$; $M_{\mathrm{B}_w} = 0.3$; $\bar{M}_{\mathrm{CL}} = 0.35$; grid $121 \times 161 \times 81$; up to a very short observation-time $t^+_{\mathrm{OBS}} = 82$).**

The evolution of the averages, for $\overline{p'u_i'}$ (Fig. 7) and for $\phi_{ij}$ Fig. 8), with $t^+_{\rm OBS}$, illustrates the convergence of the procedure.

## VI. Compressible Flow Turbulent Correlations

The compressible flow investigation concerns a ($Re_{\tau_w} = 239$; $M_{{\rm B}_w} = 1.5$; $\bar{M}_{\rm CL} = 1.5$) flow (Fig. 9). The complete splitting (10 terms) is applied to these computations, and converged results are expected within the year.

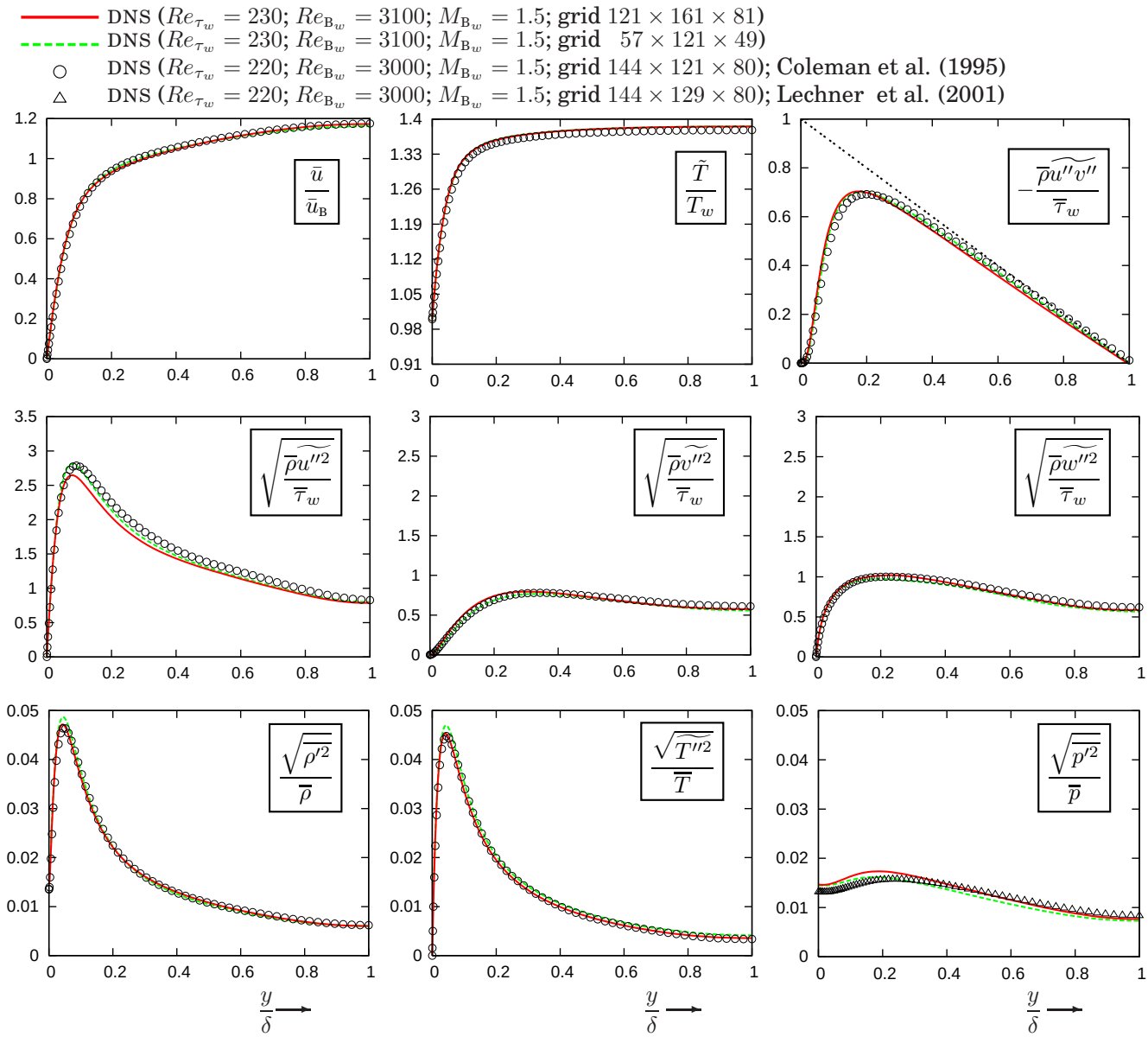


**Figure 9. Comparison of computed turbulence statistics from present DNS computations, with DNS data from Coleman et al.[37,38] and from Lechner et al.,[56] for the turbulent compressible channel flow of Coleman et al.[37,38,56] ($Re_{{\rm B}_w} = 3100$, $M_{{\rm B}_w} = 1.5$, $\bar{M}_{\rm CL} = 1.5$, isothermal walls).**

## VII. Conclusions

In the present work we extended the standard splitting of $p'$ postprocessing of DNS computations to identify seperately the volume and the wall-echo terms. The procedure uses the freespace Green's function for the evaluation of the volume part, and Kim's Green function for the complete terms, the difference between these two being the wall-echo contribution. Comparison of the various terms for $\overline{p'u_i'}$ and for $\phi_{ij}$, indicates the

importance of the wall-echo terms, but also suggests that present modelling practices with widely different homogeneous and inhomogeneous terms should be reconsidered.

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