The purpose of the present paper is to develop an efficient DNS solver for wall-bounded compressible turbulent flows. Space discretization uses high-order upwind (UW16,17 or WENO13,17) reconstruction of the primitive variables. We study in particular the UW11 schemes. The high-order upwind schemes (UW7, UW9, UW11; where UW denotes $O(\Delta x^r)$ accuracy on homogeneous grids) have good

I. Introduction

Substantial progress has been achieved in the direct numerical simulation DNS of wall-bounded compressible flows in the past decade.1–7 Initial work1,6 using pseudospectral schemes8,9 was limited to flows with 2 homogeneous directions, without shock-waves, and has therefore been replaced by research on high-order low-diffusion methods in the physical space.10–15

Most of these methods (Tab. 1) can be described by 1) the high-order reconstruction process (upwing [UW],16 WENO13,17 or monotonicity-preserving upwind schemes [MPUW],16,17) applied either on the variables13,18 or on the fluxes,10,14,15 and 2) the exact19 or approximate13,18,20 Riemann solver ARS used. Although a lot of work has been done on the study of the discretization error associated with different high-order schemes14,15,21–23 the influence of the approximate Riemann solver used ARS has received less attention. As far as time-integration is concerned, most solvers use Runge-Kutta explicit schemes, with the exception of Martin and Candler24 who use LUSGS.25,26

Table 1. Summary of physical-space high-order numerical methods applied to DNS of wall-bounded compressible flows.

<table>
<thead>
<tr>
<th>author</th>
<th>year</th>
<th>type</th>
<th>reconstruction procedure</th>
<th>Riemann solver</th>
<th>other method</th>
<th>application</th>
</tr>
</thead>
<tbody>
<tr>
<td>Adams and Shariff10</td>
<td>1996</td>
<td>flux</td>
<td>ROE–LF</td>
<td></td>
<td>entropy-fix</td>
<td>compressible channel</td>
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<td>Sesterhen11</td>
<td>2001</td>
<td>—</td>
<td>—</td>
<td>Riemann solver</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sandham et al.12</td>
<td>2002</td>
<td>—</td>
<td>—</td>
<td>Riemann solver</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pirozzoli13</td>
<td>2002</td>
<td>MUSCL</td>
<td>hybrid compact LF</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Martin et al.14</td>
<td>2006</td>
<td>flux</td>
<td>UW7/WENO</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

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spectral properties compared to other schemes, and maintain the upwind character which, as noted by Lechner et al., "generate the small amount of dissipation which is needed to suppress numerical instabilities that may be caused by unresolved high-wavenumbers." Several approximate Riemann solvers are compared. Time-integration uses an original implicit dual-time-stepping DTS scheme with explicit subiterations. A theoretical study of the complete discretization is presented.

II. Numerical Method

A. Flow Model

The flow is modelled by the compressible Navier-Stokes equations, with a spatially constant volume-force \( p_f \), in the streamwise direction, which, following Coleman et al., is adjusted to counteract viscous friction, so that a fully established compressible channel flow \( (\partial p/\partial x = 0, \partial \rho/\partial x = 0, \partial u_i/\partial x = 0) \) is obtained

\[
\frac{\partial p}{\partial t} + \frac{\partial}{\partial x_i} [p u_i] = 0
\]

(1)

\[
\frac{\partial p}{\partial t} + \frac{\partial}{\partial x_i} [p u_i u_i] = -\frac{\partial p}{\partial x_i} + \rho f_{v_i}
\]

(2)

\[
\frac{\partial}{\partial t} [\rho h - p] + \frac{\partial}{\partial x_i} [\rho h u_i] = \frac{\partial}{\partial x_i} [\mu_m \tau_{m\ell} - q] + \rho f_{v_m} u_m
\]

(3)

where \( t \) is the time, \( x_i \) are the cartesian space coordinates, \( u_i \) are the velocity components, \( \rho \) is the density, \( p \) is the pressure, \( h \) is the enthalpy, \( h_i = h + \frac{1}{2} u_i u_i \) is the total enthalpy, \( \tau_{ij} \) is the viscous-stress-tensor, and \( q_i \) is the molecular heat-flux. Perfect gas thermodynamics with constant \( c_p \) were assumed

\[
p = \rho R_g T \quad ; \quad R_g = \text{const} \quad ; \quad c_p = \frac{\gamma}{\gamma - 1} R_g = \text{const}
\]

(4)

where \( T \) is the temperature, \( R_g \) is the gas-constant, \( \gamma \) is the isentropic exponent and \( c_p \) is the heat-capacity at constant pressure. Standard linear laws were used for the constitutive relation for the stress-tensor and the heat-flux vector

\[
\tau_{ij} = \mu(T) \left[ \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \frac{\partial u_k}{\partial x_k} \delta_{ij} \right] + \mu_b \frac{\partial u_i}{\partial x_j} \delta_{ij} \equiv 2 \mu(T) \left( S_{ij} - \frac{1}{3} \delta_{ij} S_{kk} \right) + \mu_b \delta_{ij} \delta_{ij} \quad ; \quad q_i = -\lambda(T) \frac{\partial T}{\partial x_i}
\]

(5)

There has always been a lot of debate on bulk viscosity, and in particular on the way it should be incorporated in the Navier-Stokes equations. In the present study, bulk viscosity \( \mu_b \), which is not expected to be important in plane channel flow, was set to 0 (Stokes hypothesis). Viscosity \( \mu \) follows a Sutherland law, and heat-conductivity \( \lambda \) follows a modified Sutherland law

\[
\mu_b = 0 \quad ; \quad \mu(T) = \mu_0 \left( \frac{T}{T_{\text{po}}} \right)^{7/2} \frac{S_{\mu} + T_{\text{po}}}{S_{\mu} + T} \quad ; \quad \lambda(T) = \lambda_0 \frac{\mu(T)}{\mu_0} [1 + \lambda(T) \lambda_0 \mu(T)]
\]

(6)

For the present computations which are concerned with airflow, the various coefficients and constants are: \( R_g = 287.04 \text{ m}^2 \text{s}^{-1} \text{K}^{-1} \), \( \gamma = 1.4 \), \( \mu_0 \equiv \mu(T_{\text{po}}) = 17.11 \times 10^{-6} \text{ Pa s, } T_{\text{po}} = 273.15 \text{ K, } S_{\mu} = 110.4 \text{ K, } \lambda_0 \equiv \lambda(T_{\text{po}}) = 0.0242 \text{ W m}^{-1} \text{K}^{-1} \), \( A_{\lambda} = 0.00023 \text{ K}^{-1} \) (A\( _\lambda \) was obtained by a least-squares fit of the data in Eckert and Drake).

B. Space-Discretization

Defining the vectors of \( w \) of conservative variables and \( u \) of primitive variables

\[
w = [\rho, \rho u, \rho v, \rho w, \rho h - p]^{T} \in \mathbb{R}^5 \quad ; \quad u = [\rho, u, v, w, p]^{T} \in \mathbb{R}^5
\]

(7)

and the flow equations can be written

\[
\frac{\partial w}{\partial t} + \frac{\partial F}{\partial x_i} + S = 0
\]

(8)

where \( F \) \( (F_x, F_y, F_z) \) are the combined convective \( (F_x^c) \) and diffusive \( (F_x^d) \) fluxes \( (F_x = F_x^c + F_x^d) \), and \( S \) \( (S_x, S_y, S_z) \) are the source-terms (related to the volume-forces to \( f_v \)).
The above equations (Eqs. 8) are discretized on a structured multiblock grid, using a finite-volume technique. The numerical convective fluxes \( F_{i\pm\frac{1}{2},j,k}^{CN} \) are the grid-directions \((i,j,k)\) and \( S \) are the cell-face areas of the staggered-grid-cell (control-volume) around the point \((i,j,k)\), and \( F_{i\pm\frac{1}{2},j,k}^{N} \) are the numerical fluxes

\[
\frac{d\omega_{i,j,k}}{dt} + \frac{1}{V_{i,j,k}} \left[ \right. \\
\left. + \xi S_{i+j\frac{1}{2},j,k} \xi F_{i+j\frac{1}{2},j,k}^N - \xi S_{i-j\frac{1}{2},j,k} \xi F_{i-j\frac{1}{2},j,k}^N \\
+ \eta S_{i+j\frac{1}{2},j,k} \eta F_{i+j\frac{1}{2},j,k}^N - \eta S_{i-j\frac{1}{2},j,k} \eta F_{i-j\frac{1}{2},j,k}^N \\
+ \xi S_{i,j+k\frac{1}{2}} \xi F_{i,j+k\frac{1}{2}}^N - \xi S_{i,j-k\frac{1}{2}} \xi F_{i,j-k\frac{1}{2}}^N \\
+ \eta S_{i,j+k\frac{1}{2}} \eta F_{i,j+k\frac{1}{2}}^N - \eta S_{i,j-k\frac{1}{2}} \eta F_{i,j-k\frac{1}{2}}^N \right] = \overline{S}_{i,j,k} \equiv 0
\]

(9)

where \( V_{i,j,k} \) is the control-volume, \((\xi, \eta, \zeta)\) are the grid-directions \((i,j,k)\), \( S \) are the cell-face areas of the staggered-grid-cell (control-volume) around the point \((i,j,k)\), and \( F_{i\pm\frac{1}{2},j,k}^{N} \) are the numerical fluxes

\[
\xi F_{i\pm\frac{1}{2},j,k}^{N} = [\xi F_{i\pm\frac{1}{2},j,k}^{CN} (u^L, u^R, \xi n_x, \xi n_y, \xi n_z) + \xi F_{i\pm\frac{1}{2},j,k}^{VN} \xi n_i]_{i\pm\frac{1}{2},j,k}
\]

(10)

where \( F_{i\pm\frac{1}{2},j,k}^{CN} \) are the convective numerical fluxes, \( F_{i\pm\frac{1}{2},j,k}^{VN} \) are the viscous fluxes, \( u^L \) and \( u^R \) are the MUSCL-reconstructed primitive-variables at the interface, and \([\xi n_x, \xi n_y, \xi n_z]_{i\pm\frac{1}{2},j,k} \) are the unit-normals at the \( \xi = \text{const} \) cell faces. The viscous fluxes at the staggered-grid cell-faces are given by

\[
[F_{i\pm\frac{1}{2},j,k}^{VN}]_{i\pm\frac{1}{2},j,k} = \frac{1}{2} (F_{i\pm\frac{1}{2},j,k} + F_{i\pm\frac{1}{2},j,k})
\]

(11)

This results in a centered rather low \( O(\Delta x^2) \) scheme for the viscous terms.

C. Primitive Variables UW Reconstruction

Upwind-interpolations of various orders were studied, on the primitive variables \( u \)

\[
\text{UW5 : } \bar{u}_{i\pm\frac{1}{2}}^{R} = \frac{1}{36} (2u_{i+1} - 13u_{i+2} + 47u_{i+3} + 27u_{i+4} - 3u_{i+5})
\]

(12)

\[
\text{UW7 : } \bar{u}_{i\pm\frac{1}{2}}^{R} = \frac{1}{420} (-3u_{i-1} + 25u_{i-2} - 101u_{i-3} + 319u_{i-4} + 214u_{i-5} - 38u_{i-6} + 4u_{i-7})
\]

(14)

\[
\text{UW9 : } \bar{u}_{i\pm\frac{1}{2}}^{R} = \frac{1}{2220} (4u_{i-4} - 41u_{i-5} + 199u_{i-6} - 641u_{i-7} + 1879u_{i-8} + 1375u_{i-9} - 305u_{i-10} + 55u_{i-11} - 5u_{i-12})
\]

(16)

At the computational domains boundaries (which in the case of interfaces correspond to the last phantom node), the MUSCL-order is progressively reduced (when points in the stencil are not available) down to the UW1 scheme (for nodes where the UW1 stencil points are not available, a simple extrapolation procedure is used). No limiters were used for the present application, although this is theoretically incorrect, since any higher than first-order Godunov-type scheme must be nonlinear.

D. Primitive Variables WENO Reconstruction

For treating flows with shock-waves, it is necessary to introduce nonlinearity in the stencil. This was done using WENO reconstruction. We only studied the WENO11 reconstruction in the present work.

E. HLLC Approximate Riemann Solvers

The numerical convective fluxes \( F_{i\pm\frac{1}{2},j,k}^{CN} (u^L, u^R, \xi n_x, \xi n_y, \xi n_z) \) are computed using an approximate Riemann solver (ARS) between the MUSCL-reconstructed primitive variables \( u^L \) and \( u^R \). Our choice of the HLLC solver was motivated by the clear physical explanation of the method as a Godunov-scheme (by considering integration over \([\Delta x, \Delta t]\) of the waves generated by the Riemann problem), the only approximate parameters being the wave-speeds. Considering the Euler equations alone, Batten et al. have developed a straightforward method for computing the wave-speeds. The quasi-1-D Riemann problem at the cell-face is associated with 2 genuinely nonlinear (GNL) waves, corresponding to the \( V_n \pm \alpha \) eigenvalues (where \( V_n = u, n_1 \) is the normal-to-the-cell-face velocity and \( \dot{a} \) is the speed-of-sound), and with a linearly degenerate (LD) wave, corresponding.
to the triple-eigenvalue $V_s$ (contact discontinuity). The wave-speeds are noted $S_L$ and $S_R$ for the 2 GNL-waves, and $S_*$ for the LD-wave. The widely used approximate wave-speeds introduced by Einfeldt$^{43,45,46}$ for the HLL solver$^{47}$ (which ignores the contact-discontinuity) are used for the GNL-waves

$$ S_L = \min[V_{nl} - a_L, V_{nROE} - a_{ROE}] ; \quad S_R = \max[V_{nr} + a_R, V_{nROE} + a_{ROE}] \quad (18) $$

where $(\cdot)_L$ and $(\cdot)_R$ denote left- and right-MUSCL states, and $(\cdot)_{ROE}$ denote Roe-averages$^{43,48}$ between $v^L$ and $v^R$

$$ \rho_{ROE} = \sqrt{\rho_L \rho_R} \quad (19) $$

$$ u_{ROE} = \frac{\sqrt{p_L} u_L + \sqrt{p_R} u_R}{\sqrt{p_L} + \sqrt{p_R}} \quad (20) $$

$$ h_{ROE} = \frac{\sqrt{p_L} h_L + \sqrt{p_R} h_R}{\sqrt{p_L} + \sqrt{p_R}} \quad (21) $$

$$ a_{ROE} = \sqrt{(\gamma - 1)(h_{ROE} - \frac{1}{2} V_{ROE}^2)} \quad (22) $$

Batten et al.$^{44,49,50}$ used the Einfeldt $S_L$ and $S_R$ (Eqs. 18) for the 2 GNL-waves, and developed a method for approximating the $S_*$ speed for the contact discontinuity. Toro$^{43}$ indicates that their result can be obtained by assuming that the 2 intermediate states $w^L$ and $w^R$ (separated by the contact discontinuity) have the same pressure and the same normal velocity which is also the speed of the contact-discontinuity

$$ S_* = V_{nL} = V_{nR} = \frac{\rho_L(S_L - V_{nl})V_{nl} - \rho_R(S_R - V_{nr})V_{nr} - (p_L - p_R)}{\rho_L(S_L - V_{nl}) - \rho_R(S_R - V_{nr})} \quad (23) $$

$$ p_* = p_L = p_r(V_{nl} - S_L)(V_{nl} - S_*) + p_L = p_R = p_r(V_{nr} - S_R)(V_{nr} - S_*) + p_R \quad (24) $$

With these definitions the HLLC-flux reads

$$ F^{\text{HLLC}}(w^L; n_x, n_y, n_z) = \begin{cases} 
F^C(w^L; n_x, n_y, n_z) & 0 \leq S_L \\
F^C(w^L; n_x, n_y, n_z) & S_L \leq 0 \leq S_* \\
F^C(w^L; n_x, n_y, n_z) & S_* \leq 0 \leq S_R \\
F^C(w^L; n_x, n_y, n_z) & S_R \leq 0 
\end{cases} \quad (25) $$

where $F^C(w^L; n_x, n_y, n_z) = [\rho V_n, \rho u V_n + p_n, \rho e V_n + p_n, \rho w V_n + p_n, \rho h V_n]^T$ is the convective flux, and the intermediate states $w^L$, $w^R$ are given by$^{43,44,49,50}$

$$ w^L = \frac{1}{S_L - S_*} \begin{bmatrix} 
\rho_L(S_L - V_{nl}) \\
\rho_L(S_L - V_{nl})u_L + (p_L - p_R)n_x \\
\rho_L(S_L - V_{nl})u_L + (p_L - p_R)n_y \\
\rho_L(S_L - V_{nl})u_L + (p_L - p_R)n_z \\
\rho_L(S_L - V_{nl})e_{nl} - p_LS_L + p_*S_* 
\end{bmatrix} ; \quad w^R = \frac{1}{S_R - S_*} \begin{bmatrix} 
\rho_R(S_R - V_{nr}) \\
\rho_R(S_R - V_{nr})u_R + (p_R - p_L)n_x \\
\rho_R(S_R - V_{nr})e_{nr} + p_R(S_R - p_L) + p_*S_* \\
\rho_R(S_R - V_{nr})e_{nr} + p_R(S_R - p_L) + p_*S_* 
\end{bmatrix} \quad (26) $$

An if-less construction of the expression of the HLLC-flux (Eq. 26) is given in Johnsen and Colonius.$^{18}$

### F. Time-Integration

#### 1. Implicit Discretization with Explicit Subiterations

Time-discretization follows the implicit dual-time-stepping time-integration technique developed by Chassaing et al.$^{51}$ in the context of RANS computations of flows driven by external deterministic unsteadiness. Denoting by $\mathcal{L}_{i,j,k}$ the discretized form of the space-operator (divergence and source-terms; Eq. 9), the semi-discrete equation at grid-point $(i, j, k)$ gives

$$ \frac{dw_{i,j,k}}{dt} + \mathcal{L}_{i,j,k} \equiv 0 \quad \forall \quad i, j, k \quad \iff \quad \frac{dw}{dt} + \mathcal{L}(w) \equiv 0 \quad (27) $$

where $w = [w_{1,1,1}, w_{1,1,2}, \ldots, w_{N_x,N_y,N_z}]^T \in \mathbb{R}^{5 \times N_x \times N_y \times N_z}$ and $\mathcal{L} = [\mathcal{L}_{1,1,1}, \mathcal{L}_{1,1,2}, \ldots, \mathcal{L}_{N_x,N_y,N_z}]^T \in \mathbb{R}^{5 \times N_x \times N_y \times N_z}$, are the global vectors of the unknowns and of the space-operators, respectively. The time-discretization of

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the semi-discrete scheme uses a $O(\Delta t^3)$ implicit scheme. The resulting system of nonlinear equations is solved using a dual-time-stepping iterative technique.\textsuperscript{52} Denoting $\Delta t$ the physical time-step, and $\Delta t^*_{i,j,k}$ the local-dual-pseudo-time-step at grid-point $(i,j,k)$, and defining the diagonal matrix

$$\Delta t^* = \text{diag}([\Delta t^*_{1,1,1}], (\Delta t^*_{1,1,2}), \cdots, (\Delta t^*_{N_i,N_j,N_k})), \quad \text{diag}$$

(28)

where $I_5$ denotes the $5 \times 5$ identity matrix, the numerical scheme between instants $n$ and $n+1$, for the dual subiteration $m$, can be written

$$\mathcal{R}(w_n, w_{n+1}, \mathcal{L}(\Delta t)) \equiv \frac{3^n(w_n) - 4^n(w_n) + n^{-1}(w_n)}{2\Delta t} \mathcal{L}(w_n) \equiv 0 \implies$$

$$\mathcal{R}(w_n, w_{n+1}, \mathcal{L}(\Delta t)) + \frac{3^n(w_n, w_{n+1}) - 3^n(w_n) - 3^n(w_{n+1})}{2\Delta t} + \frac{\partial \mathcal{L}}{\partial w} (w_n, w_{n+1}) \frac{m}{m+1}(w_n, w_{n+1}) \equiv 0$$

(29)

where $\mathcal{R} \in \mathbb{R}^{5 \times N_i \times N_j \times N_k}$ is the nonlinear residue which was Taylor-expanded to order-1, and $\mathcal{L}(w)$ is an approximation to the operator $\mathcal{L}(w)$, chosen so as to minimize implicit work\textsuperscript{37} (the baseline choice $\mathcal{L}(w) = \mathcal{L}$ uses the exact Jacobian in the implicit procedure).

For the DNS runs, where the physical-time-step is chosen of the same order-of-magnitude as the numerical stability limit, it is much more efficient to use explicit subiterations. In that case, we simply put $\Delta t^* \equiv \partial \mathcal{L}/\partial w = 0$, and obtain an implicit $O(\Delta t^3)$ scheme with an explicit subiterative procedure. Boundary-conditions are applied at each subiteration\textsuperscript{37} (the operator $\mathcal{B}$ is used to simplify the explicit application of boundary-conditions, including the updating of phantom-nodes at the interfaces between domains $n_D$ of the computational domain). The following subiterative time-advancement procedure is obtained

```plaintext
\text{do } n_{it} = 1, N_{it}, 1; \ n \equiv n_{it}; \ t = t_0 + n\Delta t; \ 1^{n+1}w = n^{w} \\
\text{do } m_{it} \text{ while } r_{mf} \geq r_{mf}; \ n \equiv m_{it} \\
\text{do } n_{it} = 1, N_{it}, 1 \\
\quad m^{+1,n+1}w = \mathcal{B} \{(m^{n+1}w - \Delta t^* \mathcal{R}(m^{n+1}w, n^{w}, n^{-1}w, \Delta t))\} \\
\quad \text{end do; end do; end do} \quad (30)
```

where $\mathcal{J} \in \mathbb{R}^{(5 \times N_i \times N_j \times N_k) \times (2 \times N_i \times N_j \times N_k)}$ is the identity-matrix, and

$$\Delta t^* = \frac{\Delta t^*}{\mathcal{J} + \frac{3}{2\Delta t} \Delta t^*} \equiv [\mathcal{J} + \frac{3}{2\Delta t} \Delta t^*]^{-1} \Delta t^*$$

(31)

The number of subiterations (Eqs. 30) is adjusted dynamically, at each subiteration,\textsuperscript{51} to satisfy an increment $(n^{+1}w - n^{w})$ convergence-tolerance criterion $r_{mf} \geq r_{mf}$

$$r_{mf}(n+1, n+1) = \log_{10} \left\{ \frac{10^{e_m(m+1,n+1)}}{10^{e_m(m,n+1)}} \right\}$$

(32)

$$e_m(m+1, n+1) = e_m[\mathcal{B}(m^{n+1}w - m^{n+1}w)]$$

(33)

$$e_m[\mathcal{B}(m^{n+1}w - n^{w})]$$

(34)

where $\sum$ implies summation over all the grid nodes, and the summation convention for the cartesian indices $i, j = 1, 2, 3$ is used. The relative variation of the error $e_m$ (Eq. 34) is used to define the relative variation of the increment $r_{mn}$ between successive subiterations (Eq. 32). The increment-convergence-tolerance $r_{mf} < 0$ indicates the number of digits to which the computation of the increment is converged, in the subiterative procedure.

The dual-local-time-step used in the subiterative procedure (Eq. ??) is based on a combined convective (Courant) and viscous (von Neumann) criterion\textsuperscript{53}

$$\Delta t^*_{i,j,k} = \text{CFL}^* \min \left\{ \frac{\ell_g}{V + a}, \frac{\ell_g^2}{2
\nu_{eq}} \right\}; \ \nu_{eq} = \max \left\{ \frac{4}{3} \nu, \frac{\gamma - 1}{\rho \ell_s^2} \lambda \right\}$$

(35)

where $\ell_g$ is the grid-cell-size, $V$ is the flow velocity, $a$ is the sound velocity, $\nu_{eq}$ is the equivalent diffusivity,\textsuperscript{53} and $\nu$ is the molecular kinematic viscosity.
Finally, the physical time-step $\Delta t$ is chosen from physical considerations. It defines a maximum physical CFL-number used in the computation

$$\text{CFL}_{\text{max}} = \max_{i,j,k} \frac{\text{CFL}^* \Delta t}{\Delta t^{*}_{i,j,k}}$$

(36)

The parameters controlling the numerical scheme (time-integration) are the $[\Delta t; \text{CFL}^*, r_{\text{Obj}}]$. Typical computational parameters are $[\text{CFL}^*, r_{\text{Obj}}] = [1, -3]$.

This particular time-integration procedure has amplification properties which are quite different from the Runge-Kutta methods, and is less limited in the choice of the time-step.

### III. Method Validation

#### A. $M_{\text{sw}} = 1.5$; Re<sub>e</sub> = 230

### Table 3. Numerical grids used for the turbulent compressible channel flow of Coleman et al.1,2,5,6 ($R_{\text{ew}} = 3100$, $M_{\text{sw}} = 1.5$, isothermal walls).

<table>
<thead>
<tr>
<th>Authors</th>
<th>$L_x \times L_y \times L_z$</th>
<th>$N_x \times N_y \times N_z$</th>
<th>$\Delta x^+$</th>
<th>$\Delta y_{\text{sw}}^+$</th>
<th>$\Delta z^+$</th>
<th>Scheme</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coleman et al.1,2 (1995)</td>
<td>$4\pi \delta \times 25 \times \frac{4}{3} \pi \delta$</td>
<td>$144 \times 119 \times 80$</td>
<td>19</td>
<td>0.1</td>
<td>12</td>
<td>pseudo spectral</td>
</tr>
<tr>
<td>Lechner et al.5 (2001)</td>
<td>$4\pi \delta \times 25 \times \frac{4}{3} \pi \delta$</td>
<td>$144 \times 129 \times 80$</td>
<td>19</td>
<td>0.98</td>
<td>12</td>
<td>UW9</td>
</tr>
<tr>
<td>Morinishi et al.6 (2003)</td>
<td>$4\pi \delta \times 25 \times \frac{4}{3} \pi \delta$</td>
<td>$120 \times 180 \times 120$</td>
<td>23</td>
<td>0.36</td>
<td>7.6</td>
<td>pseudo spectral</td>
</tr>
<tr>
<td>Present</td>
<td>$4\pi \delta \times 25 \times \frac{4}{3} \pi \delta$</td>
<td>$57 \times 121 \times 49$</td>
<td>52</td>
<td>0.2</td>
<td>20</td>
<td>UW9</td>
</tr>
<tr>
<td>Present</td>
<td>$4\pi \delta \times 25 \times \frac{4}{3} \pi \delta$</td>
<td>$121 \times 161 \times 81$</td>
<td>24</td>
<td>0.2</td>
<td>12</td>
<td>UW9</td>
</tr>
</tbody>
</table>

$L_x$, $L_y$, $L_z$ ($N_x$, $N_y$, $N_z$) are the dimensions (number of grid-points) of the computational domain ($x =$ streamwise, $y =$ normal-to-the-wall, $z =$ homogeneous spanwise direction); $\delta$ is the channel halfheight; $\Delta x^+$, $\Delta y_{\text{sw}}^+$, $\Delta z^+$ are the mesh-sizes in wall-units; $R_{\text{ew}} = u_\text{sw} \delta_{\text{ew}}^{-1}$; $M_{\text{sw}} = u_\text{sw} \delta_{\text{sw}}^{-1}$; $u_\text{sw}$ = bulk velocity; $\rho_{\text{sw}}$ = kinematic viscosity at the wall; $u_{\text{sw}}$ = velocity-of-sound at the wall.

As an initial test of the capability of the present numerical method to predict unsteady turbulent flows, computations were performed, using the $O(\Delta x^0_{\text{ew}})$ scheme with the HLLC ARS, for the compressible turbulent channel flow ($R_{\text{ew}} = 3100$, $M_{\text{sw}} = 1.5$, isothermal walls) of Coleman et al.1,2,5,6 (Fig. 1).

The coarsest $57 \times 121 \times 49$ grid is almost 2 times coarser than previous investigations in the homogeneous $x$ (streamwise) and $z$ (cross-stream) directions (Tab. 3). The grid generation (described in Gerolymos et al.34) uses geometric stretching near the walls ($N_j = 39$ of the $N_j = 121$ points, with ratio $r_j = 1.1$ for the present grid at each wall, giving $\Delta y_{\text{ew}}^+ = 0.2$ and 20 nodes from the wall to $y^+ = 10$), the grid nodes being equidistant in the central part of the channel. Computations were run with nondimensional physical-time-step $\Delta t_{\text{phys}} \delta^{-1} = 0.00136$ (corresponding to $\text{CFL}_{\text{max}} \sim 1.7$ and dual-time-step subiterations $[\text{CFL}^*, r_{\text{Obj}}] = [1, -3]$) (roughly 5 explicit subiterations per time step are required for a reduction of 3 orders-of-magnitude of the increment. A finer $121 \times 161 \times 81$ (Tab. 2) grid was also used, which is much closer to the recommended resolution (Tab. 3). The same time-step $\Delta t$ was used on both grids. The computed statistics (first and second moments of velocity, and variances of pressure, temperature and density) agree very well with the data digitalized in Coleman et al.1,2 and in Lechner et al.5 (Fig. 2). These statistics were obtained by space-time-averaging in the homogeneous grid-directions and over $10^5$ instants spanning a nondimensional observation time of $t_{\text{obs}} u_\text{sw} \delta^{-1} = 136$, after an initial simulation time $t_{1u_\text{sw}} \delta^{-1} \sim 200$. 

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Figure 1. Instantaneous Mach-number and \( Q \)-criterion (contours for \( Q > 0 \), and isosurfaces for \( Q \delta^2 u^2 - 0.2 \)) from present DNS computations using the UW9 scheme, for the turbulent compressible channel flow of Coleman et al.\(^{1,2,5}\) (\( Re_{\tau w} = 3100 \), \( M_{B w} = 1.5 \), isothermal walls, grid \( 57 \times 121 \times 49 \)).

Figure 2. Comparison of computed turbulence statistics from present DNS computations using the UW9 scheme, on 2 different grids (\( 57 \times 121 \times 49 \), \( 121 \times 161 \times 81 \)), with data from Coleman et al.\(^{1,2}\) and from Lechner et al.,\(^{5}\) for the turbulent compressible channel flow of Coleman et al.\(^{1,2,5}\) (\( Re_{\tau w} = 3100 \), \( M_{B w} = 1.5 \), isothermal walls, \( \tau_{obs} = 4300 \)).
Notice that, in the present simulations a Sutherland-law was used for viscosity, and a modified-Sutherland-law was used for heat-conductivity (cf §II.A), while in other simulations (Tab. 3) a power-law was used for viscosity, and a constant Prandtl-number for heat-conductivity.\cite{1,2,5} Morinishi et al.\cite{6} use a Sutherland law for viscosity and a constant Prandtl number for heat conductivity.

\section*{B. $M_{Bu} = 0.3; Re_w = 180$}

\begin{center}
\includegraphics[width=\textwidth]{figure3.png}
\end{center}

\textbf{Figure 3. Comparison of present DNS-computed statistics} ($Re_{\tau w} = 230; M_{Bu} = 0.3; M_{CL} = 0.34; \tau_{OBS} = 1500$), with the UW9 scheme on 3 different grids ($57 \times 121 \times 49, 57 \times 161 \times 49, 121 \times 161 \times 81$), with incompressible DNS results of Kim et al.\cite{55,56} ($Re_{\tau w} = 180; M_B = 0$).

Computation were then run for ($Re_{\tau w} = 180; M_B = 0.3; M_{CL} = 0.34$), and compared with the classical incompressible data of Kim et al.\cite{55,56}. We compared the results using the UW9 scheme on 3 progressively finer grids ($57 \times 121 \times 49, 57 \times 161 \times 49, 121 \times 161 \times 81$; Tab. 2; Figs. 3–8), and using the UW7, UW9, UW11, and UW11 schemes on the finer $121 \times 161 \times 81$ grid (Figs. 9–13).

As far as grid-convergence is concerned, notice that although the grid influence seems limited for the first and second moments of velocity (Figs. 3–4), it is quite pronounced for higher-order moments, or for correlations containing $p'$ (Figs. 5–8). Indeed some correlations, eg, $\bar{w}'\bar{v}'$ (Fig. 5), $\bar{p}'\bar{v}'$ (Fig. 6), and $\varepsilon_{yy}$ (Fig. 8), are not sufficiently well predicted even on the finer grid.

Considering the influence of the spatial accuracy of the reconstruction of the primitive variables (Figs. 9–13), there is notable improvement going from UW7 to UW11, especially for the difficult correlations, eg, $\bar{w}'\bar{w}'\bar{v}'$ (Fig. 10), $\bar{p}'\bar{v}'$ (Fig. 11), and $\varepsilon_{yy}$ (Fig. 13). The WENO11 scheme is nearly as accurate as the UW7 scheme, although not quite as accurate (Figs. 9–13).

Notice, however, that the observation time $t_{OBS}$, although quite sufficient for the grid-convergence studies (Figs. 3–8), is insufficient for the schemes-comparison study (Figs. 9–13), and this is obvious for some of the statistics.
Figure 4. Comparison of present DNS-computed statistics ($Re_{\tau_w} = 180; \bar{M}_B = 0.34$; grid $57 \times 121 \times 49$), with the UW9 scheme on 3 different grids ($57 \times 121 \times 49, 57 \times 161 \times 49, 121 \times 161 \times 81$), with incompressible DNS results of Kim et al.55,56 ($Re_{\tau_w} = 180; \bar{M}_B = 0$).

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Figure 5. Comparison of present DNS-computed statistics \((Re_{\tau_w} = 180; M_{\theta} = 0.3; \bar{M}_{CL} = 0.34; t_{obs} = 1500)\), with the UWD scheme on 3 different grids \((57 \times 121 \times 49, 57 \times 161 \times 49, 121 \times 161 \times 81)\), with incompressible DNS results of Kim et al.\cite{Kim1980,Kim1980b} \((Re_{\tau_w} = 180; M_{\theta} = 0.3)\).
Figure 6. Comparison of present DNS-computed statistics \( (Re_{\tau_w} = 180; M_B = 0.3; \bar{M}_{CL} = 0.34; t_{obs} = 1500) \), with the UW9 scheme on 3 different grids \((57 \times 121 \times 49, 57 \times 161 \times 49, 121 \times 161 \times 81)\), with incompressible DNS results of Kim et al.\(^{55,56}\) \((Re_{\tau_w} = 180; M_B \equiv 0)\).
Figure 7. Comparison of present DNS-computed statistics ($Re_{\tau_w} = 180$; $M_B = 0.3$; $\bar{M}_{CL} = 0.34$; $t_{obs} = 1500$), with the UW9 scheme on 3 different grids ($57 \times 121 \times 49$, $57 \times 161 \times 49$, $121 \times 161 \times 81$), with incompressible DNS results of Kim et al.\textsuperscript{55,56} ($Re_{\tau_w} = 180$; $M_B \equiv 0$).
Figure 8. Comparison of present DNS-computed statistics \((Re_{\tau_w} = 180; M_B = 0.3; \bar{M}_{CL} = 0.34; t_{obs} = 1500)\), with the UWD scheme on 3 different grids \((57 \times 121 \times 49, 57 \times 161 \times 49, 121 \times 161 \times 81)\), with incompressible DNS results of Kim et al.\(^{55,56}\) \((Re_{\tau_w} = 180; M_B \equiv 0)\).
Figure 9. Comparison of present DNS-computed statistics \((Re_{\tau w} = 180; Re_{\tau v} = 2785; M_{Bw} = 0.3; \bar{M}_{CL} = 0.34),\) with the UW\textsuperscript{7} \((t_{\text{obs}} = 460),\) UW\textsuperscript{9} \((t_{\text{obs}} = 1500),\) UW\textsuperscript{11} \((t_{\text{obs}} = 440),\) and WENO\textsuperscript{11} \((t_{\text{obs}} = 160)\) schemes \((\text{grid } 121 \times 161 \times 81),\) with incompressible DNS results of Kim et al.\textsuperscript{55,56} \((Re_{\tau w} = 180; M_{B} \equiv 0).\)
Figure 10. Comparison of present DNS-computed statistics ($Re_{\tau_w} = 180$; $Re_{B_w} = 2785$; $M_{B_w} = 0.3$; grid $121 \times 161 \times 81$), with the UW11 ($t_{\text{obs}}^+ = 460$), UW9 ($t_{\text{obs}}^+ = 1500$), UW11 ($t_{\text{obs}}^+ = 140$), and WENO11 ($t_{\text{obs}}^+ = 160$) schemes (grid $121 \times 161 \times 81$), with incompressible DNS results of Kim et al.\textsuperscript{55,56} ($Re_{\tau_w} = 180$; $M_B \equiv 0$).
Figure 11. Comparison of present DNS-computed statistics ($Re_{\tau_w} = 180$; $M_B = 0.3$; $M_{CL} = 0.34$), with the UW7 ($t_{\text{obs}}^+=460$), UW9 ($t_{\text{obs}}^+=1500$), UW11 ($t_{\text{obs}}^+=440$), and WENO11 ($t_{\text{obs}}^+=160$) schemes (grid $121 \times 161 \times 81$), with incompressible DNS results of Kim et al.\textsuperscript{55,56} ($Re_{\tau_w} = 180$; $M_B \equiv 0$).
Figure 12. Comparison of present DNS-computed statistics ($Re_{\tau_w} = 180$; $M_B = 0.3$; $\bar{M}_{CL} = 0.34$), with the UW7 ($t_{OBS}^* = 460$), UW9 ($t_{OBS}^* = 1500$), UW11 ($t_{OBS}^* = 440$), and WENO11 ($t_{OBS}^* = 160$) schemes (grid $121 \times 161 \times 81$), with incompressible DNS results of Kim et al.\textsuperscript{55,56} ($Re_{\tau_w} = 180$; $M_B = 0$).
Figure 13. Comparison of present DNS-computed statistics \((Re_{\tau_w} = 180; M_{Bw} = 0.3; \bar{M}_{CL} = 0.34)\), with the UW7 \(t_{\text{obs}}^+ = 460\), UW9 \(t_{\text{obs}}^+ = 1500\), UW11 \(t_{\text{obs}}^+ = 140\), and WENO11 \(t_{\text{obs}}^+ = 160\) schemes (grid \(121 \times 161 \times 81\)), with incompressible DNS results of Kim et al.\(^{55,56}\) \((Re_{\tau_w} = 180; M_B = 0\)).
IV. Conclusions

The present computations demonstrate that the UW11 scheme offers good resolution properties for wall-bounded compressible flows, while the WENO11 scheme has resolution properties slightly inferior to the UW7 scheme, although further convergence of the statistics is necessary to fully substantiate this conclusion.

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