The Riemann Problem for Reynolds-Stress-Transport in RANS and VLES

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Summary. The present paper examines the Riemann problem for the numerical solution of the Reynolds-averaged Navier-Stokes equations with Reynolds-stress closure. Considering both the conservative convective fluxes and the Reynolds-stress production terms the Riemann problem presents 6 states, separated by waves. An HLLC–RSM approximate Riemann solver is developed.

1 Introduction

Recent work on the numerical computation of the Navier-Stokes equations with Reynolds-stress model (RSM) 7-equation turbulence closures, both in a Reynolds-averaged (RSM–RANS) framework, or in continuous RANS-to-DNS RSM–VLES approaches [1], has produced numerical methods which allow the evaluation and improvement of these advanced closures. However, especially when RSM–VLES approaches are concerned, it is important to use low-diffusion [2] high-order schemes.

The use of low-diffusion approximate Riemann solvers (ARSs) using a passive-scalar approach for the Reynolds-stresses [3, 4] fails even for simple subsonic flows (Fig. 1). In a recent work [5] this was related to the incorrect treatment of the contact discontinuity. In a classic low-diffusion solver, the massflux has no dissipation for a stationnary contact discontinuity [2] and treating the Reynolds-stresses as passive scalars yields the incorrect condition that pressure is continuous across a contact discontinuity, whereas the correct conditon is that $\bar{p} + \bar{\rho} r_n n$ should be continuous. One solution is using a hybrid scheme, with a dissipative massflux for the Reynolds-stress transport equations [5]. This solution can be used with a variety of ARSs for the meanflow equations, and yields robust schemes applicable to complex flows [5].

In the present work we examine the Riemann problem for RST, and develop an HLLC–RSM flux for the coupled system of equations.

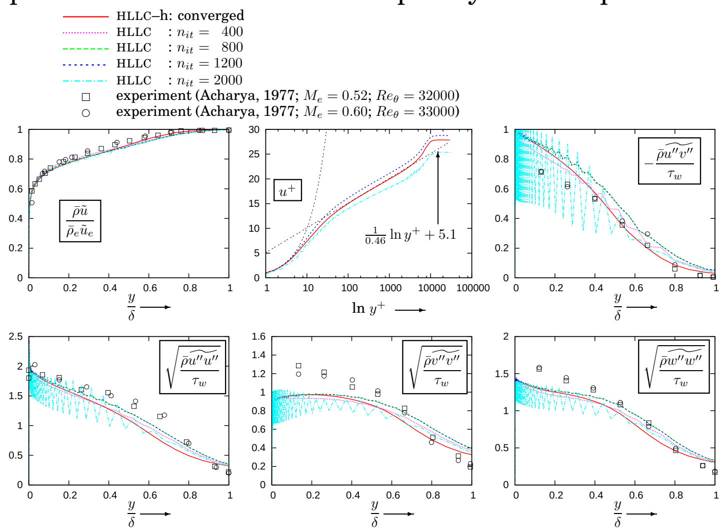

Fig. 1. Mean-mass-flux $\bar{\rho}\tilde{u}$, logarithmic law u^+, and Reynolds-stresses for near-zero-pressure-gradient boundary-layer flow, using the HLLC ARS with the passive-scalar approach for the Reynolds-stresses (comparison with measurements of Acharya[6] at $M_e = 0.22$; $Re_\theta = 21000$, and at $M_e = 0.6$; $Re_\theta = 33000$).

2 Reynolds-Stress Transport

2.1 The Complete Set of Equations

The equations are separated into a convective part (time-derivatives and first-derivatives), a diffusive part $\underline{D}$ (second-derivatives), and source-terms $\underline{S}$ (which do not contain derivatives, or are modelled terms such as the rapid part of redistribution).

$$\frac{\partial}{\partial t}\begin{bmatrix}\bar{\rho}\\ \bar{\rho}\tilde{u}_i\\ \bar{\rho}\tilde{h}_t-\bar{p}\\ \bar{\rho}r_{ij}\\ \bar{\rho}\varepsilon_u\end{bmatrix}+\frac{\partial}{\partial x_\ell}\begin{bmatrix}\bar{\rho}\tilde{u}_i\\ \bar{\rho}\tilde{u}_i\tilde{u}_\ell+\bar{p}\delta_{i\ell}+\bar{\rho}r_{i\ell}\\ \bar{\rho}\tilde{h}_t\tilde{u}_\ell+\bar{\rho}r_{k\ell}\tilde{u}_k\\ \bar{\rho}r_{ij}\tilde{u}_\ell\\ \bar{\rho}\varepsilon_u\tilde{u}_\ell\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 0\\ -P_{ij}\\ 0\end{bmatrix}=\underline{D}+\underline{S} \quad (1)$$

where the red terms correspond to the coupling of the Reynolds-stress with the meanflow equations (through the convective fluxes), while the green terms correspond to nonconservative terms coming form $-P_{ij} = \bar{\rho}r_{i\ell}\partial_{x_\ell}\tilde{u}_j + \bar{\rho}r_{j\ell}\partial_{x_\ell}\tilde{u}_i$ (exact terms)

2.2 Eigenvalues and Eigenvectors

Retaining the production terms in the RST, the system can be recast in matrix form

$$\frac{\partial \underline{v}}{\partial t} + \underline{\underline{A}}_{\ell} \frac{\partial \underline{v}}{\partial x_{\ell}} = 0 \tag{2}$$

where $\underline{\underline{A}}_{\ell} \in \mathbb{R}^{12\times 12}$ are nonstrictly hyperbolic matrices (12 real eigenvalues with multiplicity) which are not Jacobians of a flux-vector (nonconservative system). Considering, without loss of generality $\underline{\underline{A}}_{x}$

$$\underline{\underline{Ax}} = \begin{pmatrix} \tilde{u} & \bar{\rho} & 0 & 0 & 0\,0\,0\,0\,0\,0\,0\,0 \\ \frac{r_{xx}}{\bar{\rho}} & \tilde{u} & 0 & 0 & \frac{1}{\bar{\rho}}\,1\,0\,0\,0\,0\,0\,0 \\ \frac{r_{yx}}{\bar{\rho}} & 0 & \tilde{u} & 0 & 0\,0\,1\,0\,0\,0\,0\,0 \\ \frac{r_{zx}}{\bar{\rho}} & 0 & 0 & \tilde{u} & 0\,0\,0\,0\,0\,0\,1\,0 \\ 0 & \gamma\bar{p} & 0 & 0 & \tilde{u}\,0\,0\,0\,0\,0\,0\,0 \\ 0 & 2r_{xx} & 0 & 0 & 0\,\tilde{u}\,0\,0\,0\,0\,0\,0 \\ 0 & r_{yx} & r_{xx} & 0 & 0\,0\,\tilde{u}\,0\,0\,0\,0\,0 \\ 0 & 0 & 2r_{yx} & 0 & 0\,0\,0\,\tilde{u}\,0\,0\,0\,0 \\ 0 & 0 & r_{zx} & r_{yx} & 0\,0\,0\,0\,\tilde{u}\,0\,0\,0 \\ 0 & 0 & 0 & 2r_{zx} & 0\,0\,0\,0\,0\,\tilde{u}\,0\,0 \\ 0 & r_{zx} & 0 & r_{xx} & 0\,0\,0\,0\,0\,0\,\tilde{u}\,0 \\ 0 & 0 & 0 & 0 & 0\,0\,0\,0\,0\,0\,0\,\tilde{u} \end{pmatrix} \quad ; \quad \underline{v} = \begin{pmatrix} \bar{\rho} \\ \tilde{u} \\ \tilde{v} \\ \tilde{w} \\ \bar{p} \\ r_{xx} \\ r_{xy} \\ r_{yy} \\ r_{yz} \\ r_{zz} \\ r_{zx} \\ \varepsilon* \end{pmatrix} \tag{3}$$

it is straightforward to show that the eigenvalues are

$$\lambda_{\mathrm{L}} = \tilde{u} - \sqrt{\breve{a}^2 + 3r_{xx}} \tag{4a}$$

$$\lambda_{\mathrm{L}*} = \tilde{u} - \sqrt{r_{xx}} \tag{4b}$$

$$\lambda_{*} = \tilde{u} + \tag{4c}$$

$$\lambda_{\mathrm{R}*} = \tilde{u} + \sqrt{r_{xx}} \tag{4d}$$

$$\lambda_{\mathrm{L}} = \tilde{u} + \sqrt{\breve{a}^2 + 3r_{xx}} \tag{4e}$$

The analysis of a similar reduced problem by Berthon et al. [7] reveals that the Riemann problem solution for this system is composed by 2 genuinely nonlinear (GNL) waves and 3 linearly degenerate contact discontinuities (Fig. 2), thus containing 6 instead of 4 possible states for the construction of an HLLC-type flux.

2.3 Approximate Jump Relations

The nonconservative products are treated by connecting states across the dsicontinuity with a linear path [7]. In that case, the jump relations across a discontinuity with speed S, separating states 1 and 2, read

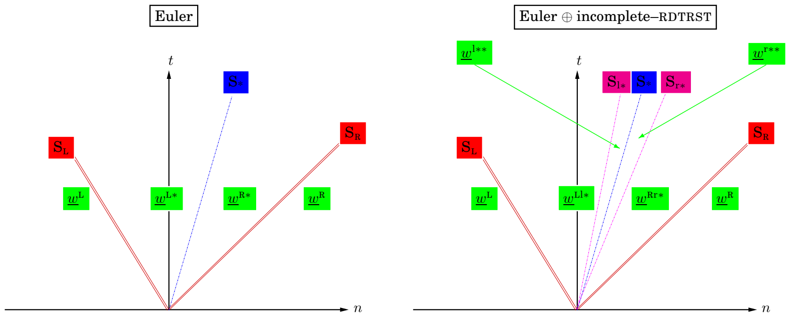

Fig. 2. Riemann problem wave system for the Euler equations[4] (2 GNL-waves and 1 LD contact discontinuity) and for the coupled Euler/RSM equations[7] (2 GNL-waves and 3 LD contact discontinuities).

$$(\mathrm{S}-\tilde{u}_2)\bar{\rho}_2 = (\mathrm{S}-\tilde{u}_1)\bar{\rho}_1 \tag{5a}$$

$$(\mathrm{S}-\tilde{u}_1)\bar{\rho}_1\Delta\tilde{u} = \Delta\bar{p} + \Delta[\bar{\rho}r_{xx}] \tag{5b}$$

$$(\mathrm{S}-\tilde{u}_1)\bar{\rho}_1\Delta\tilde{v} = \Delta[\bar{\rho}r_{xy}] \tag{5c}$$

$$(\mathrm{S}-\tilde{u}_1)\bar{\rho}_1\Delta\tilde{w} = \Delta[\bar{\rho}r_{zx}] \tag{5d}$$

$$(\mathrm{S}-\tilde{u}_1-\Delta\tilde{u})\Delta[\bar{\rho}r_{xx}] = \bar{\rho}_1 r_{xx_1}\Delta\tilde{u} + (2\bar{\rho}_1 r_{xx_1} + \Delta[\bar{\rho}r_{xx}])\Delta\tilde{u} \tag{5e}$$

$$(\mathrm{S}-\tilde{u}_1-\Delta\tilde{u})\Delta[\bar{\rho}r_{xy}] = \bar{\rho}_1 r_{xy_1}\Delta\tilde{u} + (\bar{\rho}_1 r_{xy_1} + \tfrac{1}{2}\Delta[\bar{\rho}r_{xy}])\Delta\tilde{u} + (\bar{\rho}_1 r_{xx_1} + \tfrac{1}{2}\Delta[\bar{\rho}r_{xx}])\Delta\tilde{v} \tag{5f}$$

$$(\mathrm{S}-\tilde{u}_1-\Delta\tilde{u})\Delta[\bar{\rho}r_{yy}] = \bar{\rho}_1 r_{yy_1}\Delta\tilde{u} + (2\bar{\rho}_1 r_{xy_1} + \Delta[\bar{\rho}r_{xy}])\Delta\tilde{v} \tag{5g}$$

$$(\mathrm{S}-\tilde{u}_1-\Delta\tilde{u})\Delta[\bar{\rho}r_{yz}] = \bar{\rho}_1 r_{yz_1}\Delta\tilde{u} + (\bar{\rho}_1 r_{xy_1} + \tfrac{1}{2}\Delta[\bar{\rho}r_{xy}])\Delta\tilde{w} + (\bar{\rho}_1 r_{zx_1} + \tfrac{1}{2}\Delta[\bar{\rho}r_{zx}])\Delta\tilde{v} \tag{5h}$$

$$(\mathrm{S}-\tilde{u}_1-\Delta\tilde{u})\Delta[\bar{\rho}r_{zz}] = \bar{\rho}_1 r_{zz_1}\Delta\tilde{u} + (2\bar{\rho}_1 r_{zx_1} + \Delta[\bar{\rho}r_{zx}])\Delta\tilde{w} \tag{5i}$$

$$(\mathrm{S}-\tilde{u}_1-\Delta\tilde{u})\Delta[\bar{\rho}r_{zx}] = \bar{\rho}_1 r_{zx_1}\Delta\tilde{u} + (\bar{\rho}_1 r_{zx_1} + \tfrac{1}{2}\Delta[\bar{\rho}r_{zx}])\Delta\tilde{u} + (\bar{\rho}_1 r_{xx_1} + \tfrac{1}{2}\Delta[\bar{\rho}r_{xx}])\Delta\tilde{w} \tag{5j}$$

$$(\mathrm{S}-\tilde{u}_2)\bar{\rho}_2\varepsilon^*_2 = (\mathrm{S}-\tilde{u}_1)\bar{\rho}_1\varepsilon^*_1 \tag{5k}$$

Also, obviously (Eq. 5k) ε^* is a passive scalar for this system of equations.

2.4 Approximate Jump Relations for $\lambda \neq \tilde{u}$

A straightforward solution can be obtained for $\Delta[\bar{\rho}r_{xx}]$ from (Eq. 5e), for $\lambda \neq \tilde{u}$,

$$\Delta[\bar{\rho}r_{xx}] = \frac{3\bar{\rho}_1 r_{xx_1}\Delta\tilde{u}}{S-\tilde{u}_1-2\Delta\tilde{u}} \tag{6a}$$

Approximate Jump Relations for $\lambda = \tilde{u} \pm \sqrt{\breve{a}^2 + 3r_{xx}}$

In that case all the other approximate jump-relations (Eqs. 5) can be expressed as functions of $\Delta\tilde{u}$

$$\Delta\tilde{v} = \frac{2\bar{\rho}_1 r_{xy_1} \Delta\tilde{u}}{(S - \tilde{u}_1 - \frac{3}{2}\Delta\tilde{u})(S - \tilde{u}_1)\bar{\rho}_1 - (\bar{\rho}_1 r_{xx_1} + \frac{1}{2}\Delta[\bar{\rho} r_{xx}])} \tag{7a}$$

Approximate Jump Relations for $\lambda = \tilde{u} \pm \sqrt{r_{xx}}$

In this case (LD-wave) it is reasonable to assume that

$$\mathrm{S} = \tilde{u}_1 \pm \sqrt{r_{xx_1}} = \tilde{u}_2 \pm \sqrt{r_{xx_2}} \tag{8}$$

Using these relations (Eqs. 8), in conjunction with the density jump relation (Eq. 5a), in the equations for $\Delta[\rho r_{xx}]$ (Eq. 6a), $\Delta\tilde{u}$ (Eq. **??**), and in the x-momentum jump-relation (Eq. 5b) yields

$$r_{xx_1} = r_{xx_2} \tag{9a}$$
$$\tilde{u}_1 = \tilde{u}_2 \tag{9b}$$
$$\rho_1 = \rho_2 \tag{9c}$$
$$\rho_1 r_{xx_1} = \rho_2 r_{xx_2} \tag{9d}$$
$$p_1 = p_2 \tag{9e}$$

$$\pm\bar{\rho}_1 \sqrt{r_{xx_1}} \Delta\tilde{v} = \Delta[\rho r_{xy}] \tag{10a}$$
$$\pm\bar{\rho}_1 \sqrt{r_{xx_1}} \Delta\tilde{w} = \Delta[\rho r_{zx}] \tag{10b}$$

2.5 Approximate Jump Relations for $\lambda = \tilde{u}$

On the contact discontinuity corresponding to the eigenvalue $\lambda = \tilde{u}$, it is reasonable to assume

$$\mathrm{S}_* = u_{\mathrm{L}**} = u_{\mathrm{R}**} = u_* \tag{11}$$

as in the case of the HLLC ARS for the Euler equations [4, 8]. Then the approximate jump relations become

$$\bar{p}_{\mathrm{L}**} + \bar{\rho}_{\mathrm{L}**} r_{xx_{\mathrm{L}**}} = \bar{p}_{\mathrm{R}**} + \bar{\rho}_{\mathrm{R}**} r_{xx_{\mathrm{R}**}} \tag{12a}$$
$$\bar{\rho}_{\mathrm{L}**} r_{xy_{\mathrm{L}**}} = \bar{\rho}_{\mathrm{R}**} r_{xy_{\mathrm{R}**}} \tag{12b}$$
$$\bar{\rho}_{\mathrm{L}**} r_{zx_{\mathrm{L}**}} = \bar{\rho}_{\mathrm{R}**} r_{zx_{\mathrm{R}**}} \tag{12c}$$
$$\tilde{u}_{\mathrm{L}**} = \tilde{u}_{\mathrm{R}**} \tag{12d}$$
$$\tilde{v}_{\mathrm{L}**} = \tilde{v}_{\mathrm{R}**} \tag{12e}$$
$$\tilde{w}_{\mathrm{L}**} = \tilde{w}_{\mathrm{R}**} \tag{12f}$$

2.6 Closure Relations for the HLLC–RSM Flux

Using the the jump-relations across the 2 GNL-waves we can determine the various states in the HLLC–RSM ARS, *viz*

$$\mathrm{S}_* = \frac{[\bar\rho_{\mathrm{L}}(\mathrm{S}_{\mathrm{L}} - \tilde u_{\mathrm{L}})\tilde u_{\mathrm{L}} - (\bar p + \bar\rho r_{xx})_{\mathrm{L}}] - [\bar\rho_{\mathrm{R}}(\mathrm{S}_{\mathrm{R}} - \tilde u_{\mathrm{R}})\tilde u_{\mathrm{R}} - (\bar p + \bar\rho r_{xx})_{\mathrm{R}}]}{\bar\rho_{\mathrm{L}}(\mathrm{S}_{\mathrm{L}} - \tilde u_{\mathrm{L}}) - \bar\rho_{\mathrm{R}}(\mathrm{S}_{\mathrm{R}} - \tilde u_{\mathrm{R}})} \tag{13a}$$

$$\bar\rho_{\mathrm{LL}*} r_{xx_{\mathrm{LL}*}} = \bar\rho_{\mathrm{L}**} r_{xx_{\mathrm{L}**}} = \frac{3\bar\rho_{\mathrm{L}} r_{xx_{\mathrm{L}}}(\mathrm{S}_* - \tilde u_{\mathrm{L}})}{\mathrm{S}_{\mathrm{L}} - \tilde u_{\mathrm{L}} - 2(\mathrm{S}_* - \tilde u_{\mathrm{L}})} \tag{13b}$$

$$\bar p_{\mathrm{LL}*} = \bar p_{\mathrm{L}**} = (\bar p + \bar\rho r_{xx})_{\mathrm{L}} + (S_* - \tilde u_{\mathrm{L}})\rho_{\mathrm{L}}(S_{\mathrm{L}} - \tilde u_{\mathrm{L}}) - \frac{3\bar\rho_{\mathrm{L}} r_{xx_{\mathrm{L}}}(\mathrm{S}_* - \tilde u_{\mathrm{L}})}{\mathrm{S}_{\mathrm{L}} - \tilde u_{\mathrm{L}} - 2(\mathrm{S}_* - \tilde u_{\mathrm{L}})} \tag{13c}$$

Obviously for the HLLC–RSM ARS the tangential velocities are not passive scalars. They are continuous across the $\lambda = \tilde u$ LD-wave, but not across the $\lambda = \tilde u \pm \sqrt{\breve a^2 + 3r_{xx}}$ GNL-waves nor across the $\lambda = \tilde u \pm \sqrt{r_{xx}}$ LD-waves. Using the appropriate jump relations for $\tilde v$ (Eqs. 12b, 12e, 10b, 7a) it follows that

$$\tilde v_{\mathrm{L}**} = \tilde v_{\mathrm{R}**} = \frac{\left(\bar\rho_{\mathrm{LL}*}\sqrt{r_{xx_{\mathrm{LL}*}}}\tilde v_{\mathrm{LL}*} + \bar\rho_{\mathrm{RR}*}\sqrt{r_{xx_{\mathrm{RR}*}}}\tilde v_{\mathrm{RR}*} + \bar\rho_{\mathrm{RR}*} r_{xy_{\mathrm{RR}*}} - \bar\rho_{\mathrm{LL}*} r_{xy_{\mathrm{LL}*}}\right)}{\bar\rho_{\mathrm{LL}*}\sqrt{r_{xx_{\mathrm{LL}*}}} + \bar\rho_{\mathrm{RR}*}\sqrt{r_{xx_{\mathrm{RR}*}}}} \tag{14}$$

with a similar relation for $\tilde w$. The above relations completely define the HLLC–RSM flux.

3 Conclusions

Reynolds-stresses transport cannot be accomodated, in low-diffusion (contact-discontinuity resolving) ARSs, by simply using the passive scalar approach. In the present work we developed an HLLC–RSM ARS for RST.

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